IRON SHIELDING FOR AIR CORE MAGNETS*

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INTRODUCTION

Air core magnets, whether superconducting or cryogenic, have high external fields which, in practical applications, can be troublesome. These can be removed either by additional windings (which decrease the useful field) or by using iron (which increases the useful field). This paper is on the consequences of the latter procedure.

COILS OF INFINITESIMAL THICKNESS

The problem of distribution of fields around a multipole air core magnet surrounded by iron has been solved by Beth.[1] It has been treated also by Meuser.[2] The configuration studied by these authors is shown in Fig. 1. Region (1) of radius a_1 is surrounded by a winding of infinitesimal thickness (a_2 in Fig. 1 is assumed equal to a_1) bearing a current normal to the paper of

$$I = I_o \cos n\theta \quad .$$

The region (3) beyond radius r = b is filled with iron of permeability μ.

For the present purpose we assume μ to be infinitely large and write Beth's solution in the form:

In region (1):

$$B_r = 2\pi \mu_o I_o (r/a)^{n-1} \left\{ 1 + (a/b)^{2n} \right\} \sin n\theta$$

$$B_\theta = 2\pi \mu_o I_o (r/a)^{n-1} \left\{ 1 + (a/b)^{2n} \right\} \cos n\theta \quad . \tag{1}$$

In region (2):

$$B_r = 2\pi \mu_o I_o a^{n+1} \left\{ \frac{r^{n-1}}{b^{2n}} + \frac{1}{r^{n+1}} \right\} \sin n\theta$$

$$B_\theta = 2\pi \mu_o I_o a^{n+1} \left\{ \frac{r^{n-1}}{b^{2n}} - \frac{1}{r^{n+1}} \right\} \cos n\theta \quad . \tag{2}$$

This solution gives zero azimuthal field at r = b. The maximum normal field is

$$B_{max} = 4\pi \mu_o I_o (a/b)^{n+1} \quad . \tag{3}$$

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*Work performed under the auspices of the U.S. Atomic Energy Commission.

1. R.A. Beth, Brookhaven National Laboratory, Accelerator Dept. Report AADD-119 (1966).
2. R.B. Meuser, Lawrence Radiation Laboratory Report UCRL-18318 (1968).

For the case where B_{max} is assigned a specific value B_o (to avoid saturation, for example), the expression (3) defines b in terms of B_{max} and the other parameters.

In Eqs. (1), the term $(a/b)^{2n}$ represents the increase in field inside the magnet due to the presence of the iron. From (1) and (3) the field amplitude is

$$2\pi \mu_o I_o (r/a)^{n-1} \left\{ 1 + \left(\frac{B_o}{4\pi \mu_o I_o} \right)^{2n/n+1} \right\} .$$

The second term is the increase due to the iron shield. For a dipole, (n = 1), the increase is simply $B_o/2$. If fields as high as 20 kG are allowed at r = b, the iron will add 10 kG to whatever field is set up inside the magnet. For a quadrupole, (n = 2), the additional term in the bracket is

$$\left(\frac{B_o}{4\pi \mu_o I_o} \right)^{4/3} .$$

If, for example, I_o is such as to give 60 kG "pole-tip fields" without iron and B_o is chosen to be 20 kG, the increase in pole-tip field due to the iron will be about 5600 G.

The radius b for a 60 kG dipole with 20 kG at r = b will be given approximately by

$$b = 2a \quad .$$

The radius b for a 60 kG pole-tip quadrupole with 20 kG at r = b will be given approximately by

$$b = 1.7a \quad .$$

COILS OF FINITE THICKNESS

To obtain the fields for coils of finite thickness we merely replace I_o in Eqs. (1) and (2) by $I_o da$ and integrate from an inner radius a_1 to an outer radius a_2. For radii $r < a_1$ it is necessary only to integrate Eqs. (1) to obtain

$$B_r = 2\pi \mu_o I_o r^{n-1} \left\{ \frac{1}{2-n} \left(a_2^{2-n} - a_1^{2-n} \right) + \frac{1}{(n+2)\, b^{2n}} \left(a_2^{n+2} - a_1^{n+2} \right) \right\} \sin n\theta$$

$$B_\theta = 2\pi \mu_o I_o r^{n-1} \left\{ \frac{1}{2-n} \left(a_2^{2-n} - a_1^{2-n} \right) + \frac{1}{(n+2)\, b^{2n}} \left(a_2^{n+2} - a_1^{n+2} \right) \right\} \cos n\theta \quad . \tag{4}$$

For radii $r > a_2$ we integrate Eqs. (2) to obtain:

$$B_r = 2\pi \mu_o I_o \left(\frac{r^{n-1}}{b^{2n}} + \frac{1}{r^{n+1}} \right) \left(\frac{1}{n+2} \right) \left(a_2^{n+2} - a_1^{n+2} \right) \sin n\theta$$

$$B_\theta = 2\pi \mu_o I_o \left(\frac{r^{n-1}}{b^{2n}} - \frac{1}{r^{n+1}} \right) \left(\frac{1}{n+2} \right) \left(a_2^{n+2} - a_1^{n+2} \right) \cos n\theta \quad . \tag{5}$$

Inside the coil, for $a_2 > r > a_1$, the solution is a combination of solutions of types (4) and (5):

$$B_r = 2\pi \mu_o I_o \left[r^{n-1} \left\{ \frac{a_2^{2-n}}{2-n} + \frac{1}{(n+2)\, b^{2n}} \left(a_2^{n+2} - a_1^{n+2} \right) \right\} - \frac{a_1^{n+2}}{(n+2)\, r^{n+1}} + \frac{2nr}{n^2-4} \right] \sin n\theta$$

(6)

$$B_\theta = 2\pi \mu_o I_o \left[r^{n-1} \left\{ \frac{a_2^{2-n}}{2-n} + \frac{1}{(n+2)\, b^{2n}} \left(a_2^{n+2} - a_1^{n+2} \right) \right\} + \frac{a_1^{n+2}}{(n+2)\, r^{n+1}} + \frac{4r}{n^2-4} \right] \cos n\theta \ .$$

The vector potentials in the three regions are:

for $r < a_1$

$$A_z = -2\pi \mu_o I_o \frac{r^n}{n} \left\{ \frac{1}{2-n} \left(a_2^{2-n} - a_1^{2-n} \right) + \frac{1}{(n+2)\, b^{2n}} \left(a_2^{n+2} - a_1^{n+2} \right) \right\} \cos n\theta \ , \qquad (7)$$

for $r > a_2$

$$A_z = -\frac{2\pi \mu_o I_o}{n(n+2)} \left(a_2^{n+2} - a_1^{n+2} \right) \left(\frac{r^n}{b^{2n}} + \frac{1}{r^n} \right) \cos n\theta \ , \qquad (8)$$

for $a_2 > r > a_1$

$$A_z = -2\pi \mu_o I_o \left[\frac{r^n}{n} \left\{ \frac{a_2^{2-n}}{2-n} + \frac{1}{(n+2)\, b^{2n}} \left(a_2^{n+2} - a_1^{n+2} \right) \right\} - \frac{a_1^{n+2}}{n(n+2)\, r^n} + \frac{2\, r^2}{n^2-4} \right] \cos n\theta \ .$$

(9)

DIPOLE FIELDS

For a dipole, the field patterns are:

for $r < a_1$

$$B_r = 2\pi \mu_o I_o \left\{ \left(a_2 - a_1 \right) + \frac{1}{3b^2} \left(a_2^3 - a_1^3 \right) \right\} \sin \theta$$

(10)

$$B_\theta = 2\pi \mu_o I_o \left\{ \left(a_2 - a_1 \right) + \frac{1}{3b^2} \left(a_2^3 - a_1^3 \right) \right\} \cos \theta \ ,$$

for $r > a_2$

$$B_r = (2/3)\pi \mu_o I_o \left(\frac{1}{b^2} + \frac{1}{r^2} \right) \left(a_2^3 - a_1^3 \right) \sin \theta$$

(11)

$$B_\theta = (2/3)\pi \mu_o I_o \left(\frac{1}{b^2} - \frac{1}{r^2} \right) \left(a_2^3 - a_1^3 \right) \cos \theta \ ,$$

for $a_2 > r > a_1$

$$B_r = 2\pi \mu_o I_o \left\{ a_2 + \frac{1}{3b^2}\left(a_2^3 - a_1^3 \right) - \frac{a_1^3}{3r^2} - \frac{2}{3} r \right\} \sin\theta$$

$$B_\theta = 2\pi \mu_o I_o \left\{ a_2 + \frac{1}{3b^2}\left(a_2^3 - a_1^3 \right) + \frac{a_1^3}{3r^2} - \frac{4}{3} r \right\} \cos\theta \quad . \qquad (12)$$

The vector potential for a dipole is:

for $r < a_2$

$$A_z = -\,2\pi \mu_o I_o r \left\{ \left(a_2 - a_1 \right) + \frac{1}{3b^2}\left\{ a_2^3 - a_1^3 \right) \right\} \cos\theta \quad , \qquad (13)$$

for $r > a_2$

$$A_z = -\,\frac{2}{3}\pi \mu_o I_o r \left(\frac{1}{b^2} + \frac{1}{r^2} \right)\left(a_2^3 - a_1^3 \right) \cos\theta \quad , \qquad (14)$$

for $a_2 > r > a_1$

$$A_z = -\,2\pi \mu_o I_o r \left\{ a_2 + \frac{1}{3b^2}\left(a_2^3 - a_1^3 \right) - \frac{a_1^3}{3r^2} - \frac{2}{3} r \right\} \cos\theta \quad . \qquad (15)$$

The field patterns for two possible dipoles are shown in Figs. 2 and 3. Figure 2 represents a dipole for 40 kG in an aperture of 5 cm radius. The coil is 1 cm in thickness and is surrounded by an iron shield of 10 cm radius. The maximum flux density entering the iron is 20 kG. To carry this flux at an average density of 16 kG the iron should be about 13 cm thick. Current density in the coil has a maximum value of about 6×10^4 A/cm^2.

Figure 3 represents a 60 kG magnet of the same aperture. The thickness of the coil is 1.5 cm, which results in about the same peak current density (6×10^4 A/cm^2) as the coil of Fig. 2. The iron shield has an inner radius of 12.9 cm and a peak entering flux density of 20 kG. To carry the return flux at an average density of 16 kG, the iron should be about 16 cm thick. The weight of this shield would be slightly less than 2 tons per meter of length.

QUADRUPOLE FIELDS

When n is set equal to 2 in Eqs. (4)-(9), the reader may be disturbed by the appearance of $2 - n$ in the denominator of several terms. This, however, appears in terms of the form

$$\frac{1}{2-n}\left(a_2^{2-n} - a_1^{2-n} \right) \quad .$$

As n approaches 2, set $n = 2 - \Delta$. The term now becomes

$$\frac{1}{\Delta}\left(a_2^{\Delta} - a_1^{\Delta} \right) = \frac{1}{\Delta}\left(e^{\Delta \ln a_2} - e^{\Delta \ln a_1} \right) \quad .$$

As $\Delta \to 0$, this term becomes $\ln(a_2/a_1)$. Similar treatment removes the apparent singularity wherever it appears.

The vector potential for a quadrupole is:

for $r < a_1$

$$A_z = -\pi\mu_o I_o r^2 \left\{ \ln(a_2/a_1) + \frac{1}{4b^4}\left(a_2^4 - a_1^4\right)\right\} \cos 2\theta \quad ,$$

for $r > a_2$

$$A_z = -\frac{\pi\mu_o I_o}{4}\left(a_2^4 - a_1^4\right)\left(\frac{r^2}{b^4} + \frac{1}{r^2}\right) \cos 2\theta \quad ,$$

for $a_2 > r > a_1$

$$A_z = -\pi\mu_o I_o r^2 \left\{ \ln(a_2/r) + \frac{1}{4b^4}\left(a_2^4 - a_1^4\right) - \frac{a_1^4}{4r^4} + \frac{1}{4}\right\} \cos 2\theta \quad .$$

Figure 4 shows the field patterns for a quadrupole with 60 kG "pole-tip fields" surrounded by an iron shield whose maximum entering flux density is 20 kG. For an aperture of 5 cm radius, the coil thickness is 1.5 cm. The coil, as before, carries a cos 2θ current distribution of maximum density of about 6×10^4 A/cm^2. The inner radius of the iron shield is 10.7 cm. To carry the flux at an average density of 16 kG the iron should be about 7 cm thick.

IRON CROSS SECTION

Numerical examples thus far have used the case where the flux density of the iron ring is 20 kG (2 T). In this section it will be shown that less iron can be used if lower fields are chosen at the iron surface. This will result in less increase in the fields in the multipole due to the presence of iron, but it may nevertheless have economic advantages.

From Eqs. (5) the normal field at the surface of the iron is

$$B_r = \frac{4\pi\mu_o I_o}{n+2}\frac{1}{b^{n+1}}\left(a_2^{n+2} - a_1^{n+2}\right) \sin n\theta \quad .$$

If b_2 is the iron inner radius for which a maximum normal field of 2 T is set up, then

$$2 = \frac{4\pi\mu_o I_o}{n+2}\frac{1}{b_2^{n+1}}\left(a_2^{n+2} - a_1^{n+2}\right) \quad .$$

and

$$B_r = 2\left(\frac{b_2}{b}\right)^{n+1} \sin n\theta \quad .$$

The total flux that must be carried by the iron is that entering between $\theta = 0$ and $\theta = \pi/2n$, and is

$$\int_0^{\pi/2n} B_r \, b \, d\theta = \frac{2b_2}{n} \left(\frac{b_2}{b} \right)^n .$$

If the flux returns in the iron with an average flux density B_1 and the iron ring has an outer diameter c, then

$$(c - b) \, B_1 = \frac{2b_2}{n} \left(\frac{b_2}{b} \right)^n$$

and

$$c = \frac{2b_2}{nB_1} \left(\frac{b_2}{b} \right)^n + b .$$

The cross-sectional area of the iron is

$$\pi \, (c^2 - b^2) = \frac{4\pi \, b_2^2}{nB_1} \left(\frac{b_2}{b} \right)^{n-1} \left\{ \frac{1}{nB_1} \left(\frac{b_2}{b} \right)^{n+1} + 1 \right\} .$$

For a dipole the area is

$$\frac{4\pi \, b_2^2}{B_1} \left\{ \frac{1}{B_1} \left(\frac{b_2}{b} \right)^2 + 1 \right\} .$$

For large b's the iron area approaches a constant value of $4\pi \, b_2^2/B_1$. If B_1 is 16 kG (1.6 T) as we have assumed in our previous examples, the maximum reduction in iron area possible below that necessary for 20 kG entering field is [(1/1.6) + 1]:1 or about 1.6:1.

For a quadrupole, the iron area is

$$\frac{4\pi \, b_2^3}{2B_1 \, b} \left\{ \frac{b_2^3}{2B_1 b^3} + 1 \right\} .$$

Evidently, by reducing the entering field, the necessary iron for shielding a quadrupole can be reduced as far as desired.

DISTRIBUTION OF STORED ENERGY

For an evaluation of stored energy in the various regions, an approximate method will be used. It will be assumed that the thickness of the coil is small compared with the aperture dimensions. The following notation will be used:

$$a_2 = a_1 + \Delta$$

$$r = a_1 + x \text{ (within the winding)}$$

$$2\pi \, \mu_o I_o \Delta \, [1 + (a_1/b)^{2n}] = B_o .$$

The approximation will be carried only to the first order.

Equations (4) to (6) can now be written:

for $r < a_1$

$$B_r = B_o\,(r/a_1)^{n-1}\,\sin n\theta$$

$$B_\theta = B_o\,(r/a_1)^{n-1}\,\cos n\theta \quad ,$$

for $a_1 < r < a_2$

$$B_r = B_o\,\sin n\theta$$

$$B_\theta = B_o \left\{ 1 - \frac{2x/\Delta}{1 + (a_1/b)^{2n}} \right\} \cos n\theta \quad ,$$

for $r > a_2$

$$B_r = \frac{B_o}{1 + (a_1/b)^{2n}}\,(a_1/r)^{n+1} \left[1 + (r/b)^{2n} \right] \sin n\theta$$

$$B_\theta = \frac{-\,B_o}{1 + (a_1/b)^{2n}}\,(a_1/r)^{n+1} \left[1 - (r/b)^{2n} \right] \cos n\theta \quad .$$

Using these expressions the stored energy E in the various regions can easily be evaluated:

for $r < a_1$

$$E = \frac{B_o^2\,a_1^2}{8n\,\mu_o} \quad ,$$

for $a_1 < r < a_2$

$$E = \frac{B_o^2\,a_1\,\Delta}{4\mu_o} \left\{ \frac{2/3 + (a_1/b)^{2n} + (a_1/b)^{4n}}{\left[1 + (a_1/b)^{2n} \right]^2} \right\} \quad ,$$

for $r > a_2$

$$E = \frac{B_o^2\,a_1^{2n+2}}{8n\,\mu_o\,a_2^{2n}} \left\{ \frac{1 - (a_2/b)^{4n}}{\left[1 + (a_1/b)^{2n} \right]^2} \right\} \quad .$$

If $(a_1/b)^{2n}$ is small, and $a_2 \cong a_1$, then the energy stored outside the coil is approximately the same as that stored in the useful aperture.

The average energy density in the aperture is

$$\frac{B_o^2}{8n\,\pi\,\mu_o}$$

In the coil it is

$$\frac{B_o^2}{8\pi\,\mu_o}\left\{\frac{2/3 + (a_1/b)^{2n} + (a_1/b)^{4n}}{\left[1 + (a_1/b)^{2n}\right]^2}\right\} .$$

For the particular case where $b = 2a_1$, the ratio of energy density in the coil to energy density in the useful aperture is

for a dipole:	0.63
for a quadrupole:	1.29 .

For cases of the type discussed above where $\Delta = 0.3\ a_1$ the ratio of the total energy stored in the coil to the total energy stored in the useful aperture will be

for a dipole:	0.44
for a quadrupole:	0.89 .

Hence for a dipole the total energy stored in the coil (for the dimensions quoted) is a little more than 20% of the total energy stored in the dipole system. For a quadrupole of similar dimensions the energy stored in the coil is about 45% of the total energy stored in the system.

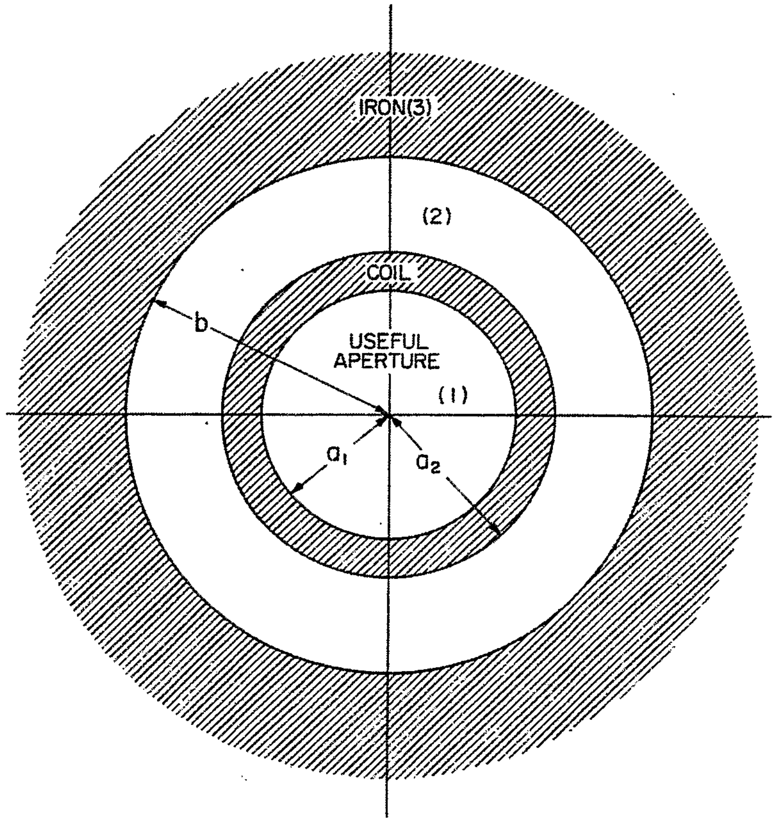

Fig. 1. Geometry of multipole winding and iron shield.

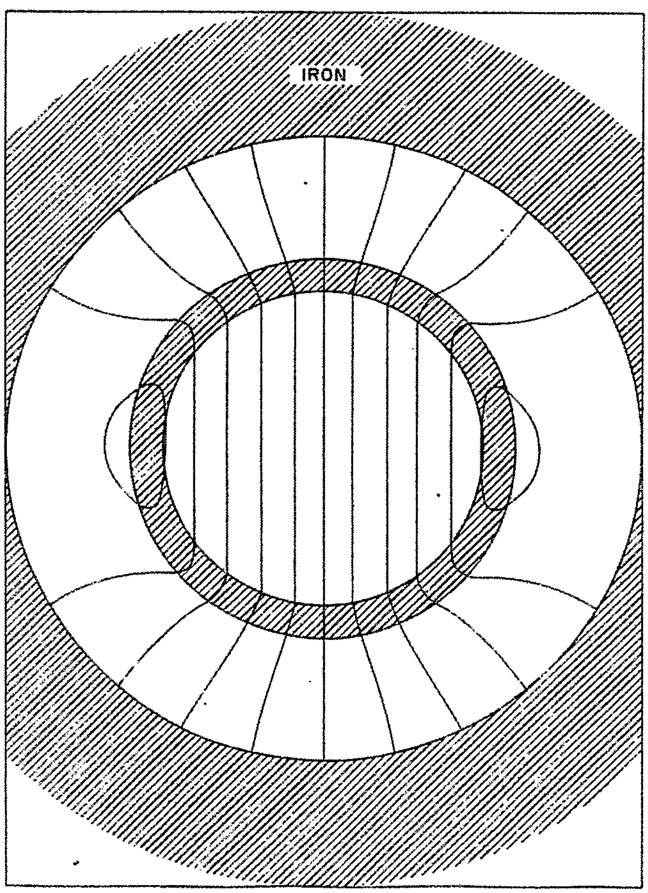

Fig. 2. Field pattern for 40 kG dipole.

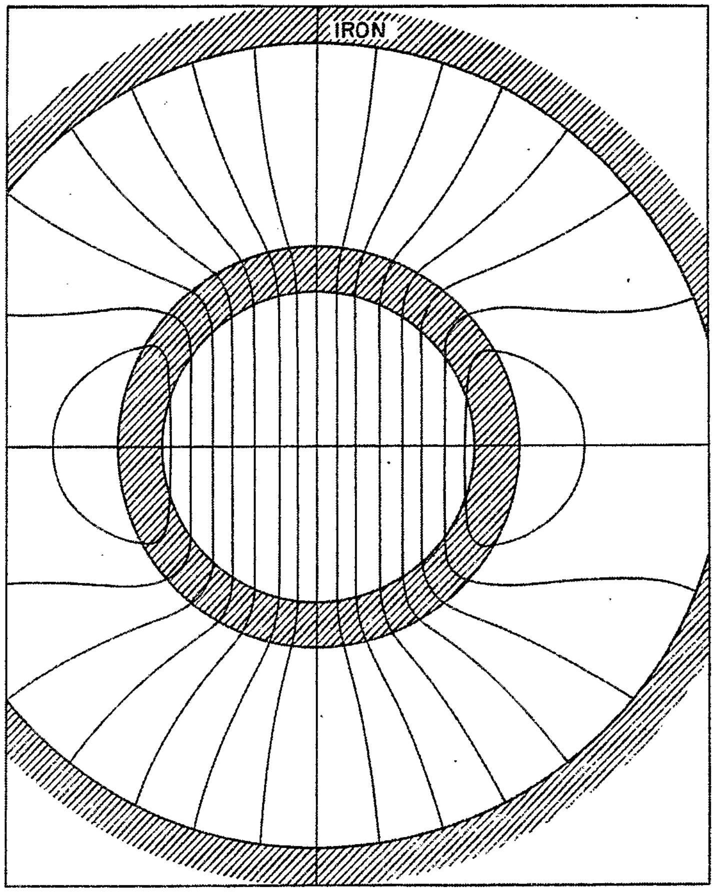

Fig. 3. Field pattern for 60 kG dipole.

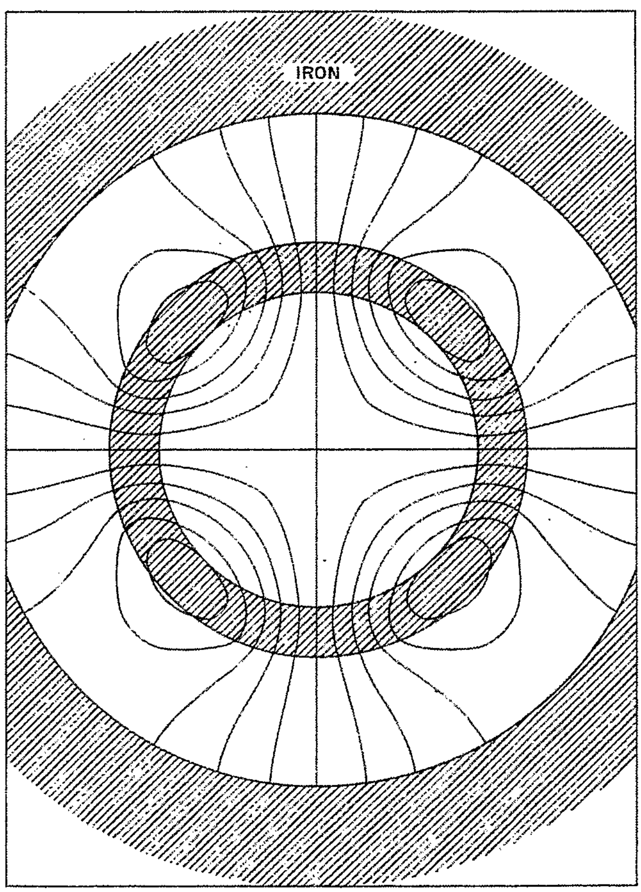

Fig. 4. Field pattern for quadrupole with 60 kG pole-tip field.