Harmonic Description of 2-Dimensional Fields

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Fields in Free Space: Scalar Potential

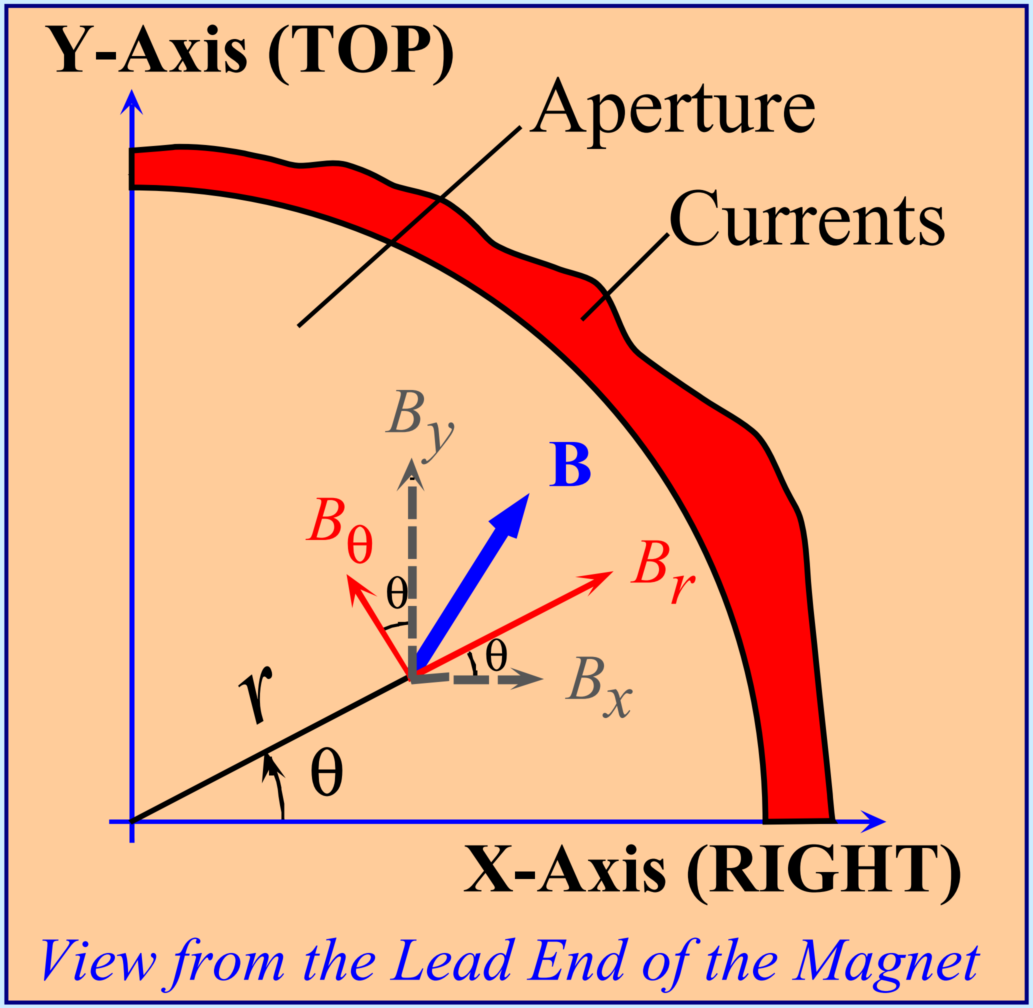
- $\nabla.\mathbf{B} = 0$ (Always true)
- In a region free of any currents or magnetic material, $\nabla \times \mathbf{B} = 0$, and **B** may be written as the gradient of a scalar potential, $\mathbf{B} = \nabla\Phi_m$
- The two equations above may be combined to obtain the Laplace's equation for the scalar potential, Φ_m,

$$\nabla^2\Phi_m = 0$$

2-D Fields in Free Space

- $\boldsymbol{B} = \nabla\Phi_m$ and $\nabla^2\Phi_m = 0$
- Most accelerator magnet apertures have a cylindrical symmetry, with a length much larger than the aperture. In such situations, the field away from the ends can be considered 2-dimensional, and the general solution can be expressed in a relatively simple ***harmonic series***.

Commonly Used Coordinate System



Users of magnetic measurements data may use a system oriented differently, often requiring suitable transformations of the measured harmonics.

$$B_x(r,\theta) = B_r \cos\theta - B_\theta \sin\theta$$

$$B_y(r,\theta) = B_r \sin\theta + B_\theta \cos\theta$$

Solution in Cylindrical Coordinates

For no z-dependence (2-D fields),

$$\nabla^2 \Phi_m = \left(\frac{1}{r}\right)\frac{\partial}{\partial r}\left(r\frac{\partial \Phi_m}{\partial r}\right) + \left(\frac{1}{r^2}\right)\left(\frac{\partial^2 \Phi_m}{\partial \theta^2}\right) = 0$$

writing $\Phi_m(r,\theta) = R(r)\Theta(\theta)$, and imposing the conditions

$$\Theta(\theta + 2\pi) = \Theta(\theta); \quad R(r) = \text{finite at } r = 0$$

we can get the solution of the Laplace's equation in terms of a harmonic series.

2-D Fields: Harmonic Series

- Components of 2-D fields in cylindrical coordinates:

$$B_r(r,\theta) = \sum_{n=1}^{\infty} C(n) \left(\frac{r}{R_{ref}} \right)^{n-1} \sin[n(\theta - \alpha_n)]$$

$$B_\theta(r,\theta) = \sum_{n=1}^{\infty} C(n) \left(\frac{r}{R_{ref}} \right)^{n-1} \cos[n(\theta - \alpha_n)]$$

- $C(n)$ = ***Amplitude***, α_n = ***phase angle*** of the ***2n-pole term*** in the expansion.
- R_{ref} = ***Reference radius***, arbitrary, typically chosen ~ the region of interest.

C(n) scales as R_{ref}^{n-1}

2-D Fields: Cartesian Components

- Cartesian components of **B** may be written as:

$$B_x(r,\theta) = \sum_{n=1}^{\infty} C(n) \left(\frac{r}{R_{ref}} \right)^{n-1} \sin[(n-1)\theta - n\alpha_n]$$

$$B_y(r,\theta) = \sum_{n=1}^{\infty} C(n) \left(\frac{r}{R_{ref}} \right)^{n-1} \cos[(n-1)\theta - n\alpha_n]$$

- A ***Complex field***, $\boldsymbol{B}(z) = B_y + iB_x$, where $z = x + iy$, combines the 2 equations above:

$$\boldsymbol{B}(z) = \sum_{n=1}^{\infty} [C(n)\exp(-in\alpha_n)] \left(\frac{z}{R_{ref}} \right)^{n-1}$$

2-D Fields: Normal & Skew Terms

$$\boldsymbol{B}(z) = B_y + iB_x = \sum_{n=1}^{\infty} [C(n)\exp(-in\alpha_n)] \left(\frac{z}{R_{ref}}\right)^{n-1}$$

may be written as:

$$\boldsymbol{B}(z) = \sum_{n=1}^{\infty} [B_n + iA_n] \left(\frac{z}{R_{ref}}\right)^{n-1}$$

Simple power series, valid within source-free zone.

where:

$$B_n \equiv C(n)\cos(n\alpha_n) = 2n\text{ - pole NORMAL Term}$$

$$A_n \equiv -C(n)\sin(n\alpha_n) = 2n\text{ - pole SKEW Term}$$

In the US, the 2*n*-pole terms are denoted by B_{n-1} and A_{n-1}.

Sometimes, the skew terms are defined without the negative sign, but the above form is the most common now.

Analytic Functions of a Complex Variable

Any function of the complex variable, z, given by

$$\boldsymbol{F}(z) = U(x,y) + i\,V(x,y)$$

is an ***Analytic*** function of z, if

$$\left(\frac{\partial U}{\partial x}\right) = \left(\frac{\partial V}{\partial y}\right) \quad \text{and} \quad \left(\frac{\partial U}{\partial y}\right) = -\left(\frac{\partial V}{\partial x}\right)$$

Cauchy-Riemann Conditions.

An analytic function can be expressed as a power series in z. This series is valid within the ***circle of convergence***, which extends to the nearest singularity. Analytic function does not depend on z^*.

Analyticity of Complex Field

$$\left(\frac{\partial U}{\partial x}\right) = \left(\frac{\partial V}{\partial y}\right) \quad \text{and} \quad \left(\frac{\partial U}{\partial y}\right) = -\left(\frac{\partial V}{\partial x}\right)$$

Cauchy-Riemann Conditions.

Maxwell's equations in source free region:

$$\nabla.\boldsymbol{B} = 0 \Rightarrow \left(\frac{\partial B_y}{\partial y}\right) = -\left(\frac{\partial B_x}{\partial x}\right)$$

$$(\nabla \times \boldsymbol{B})_z = 0 \Rightarrow \left(\frac{\partial B_y}{\partial x}\right) = \left(\frac{\partial B_x}{\partial y}\right)$$

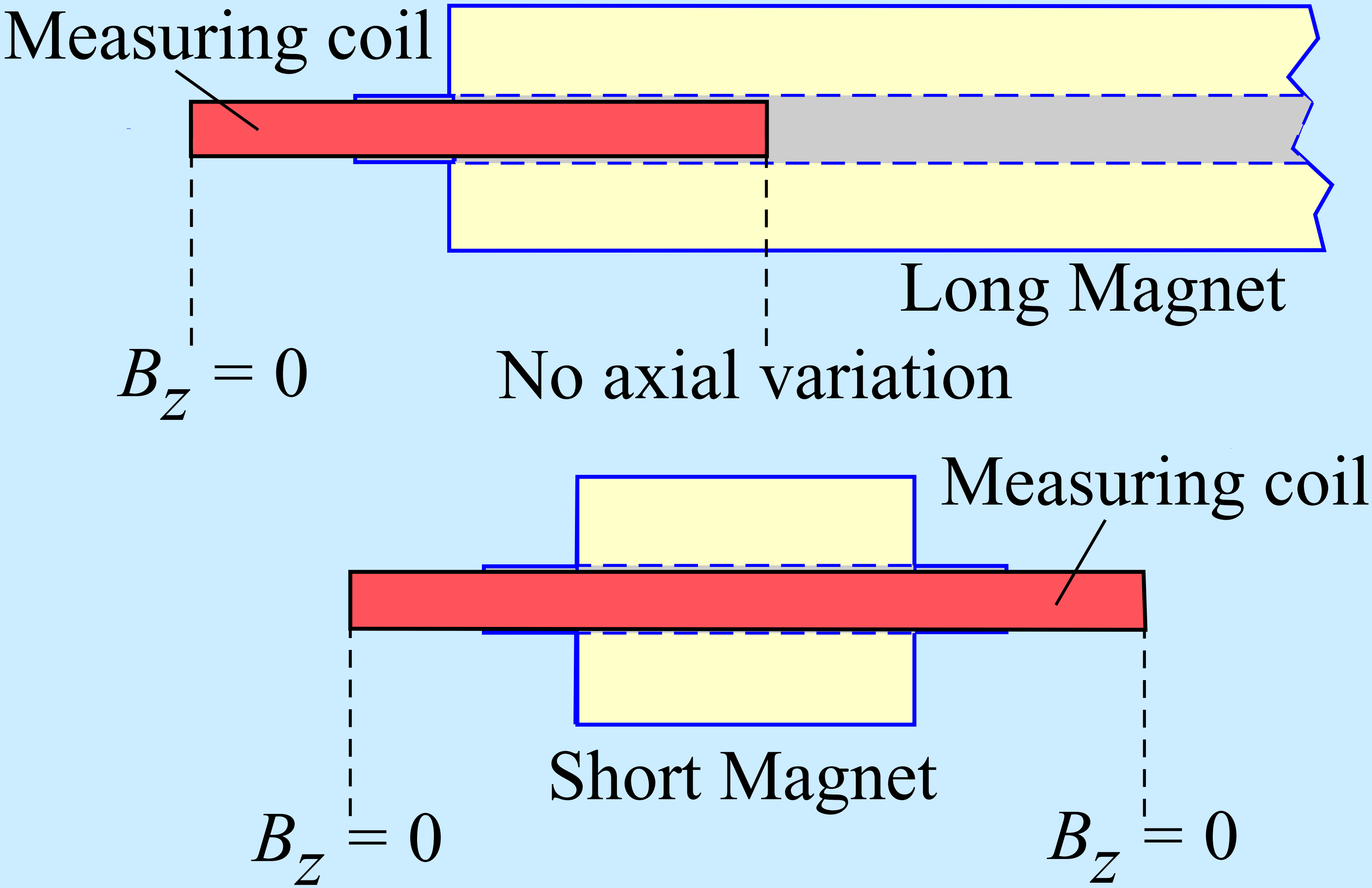
Maxwell's equations = Cauchy-Riemann conditions if we choose: $U(x,y) = B_y(x,y)$ and $V(x,y) = B_x(x,y)$

Thus, $\boldsymbol{B}(\boldsymbol{z}) = B_y(x,y) + i\, B_x(x,y)$ is an analytic function of $\boldsymbol{z}$. The analyticity is useful in dealing with 2-D problems in magnetostatics.

End Fields & Short Magnets

- The field near the ends of a long magnet, or everywhere in a short magnet, has all three components. The simple 2-D expansion is not valid in these cases. However, if one considers only integrated values of field components, a similar 2-D expansion can be shown to be valid.
- For components of field at a point, a more complex expansion must be used.

Validity of 2-D Field Expansion



3-D Field Expansion

If the field harmonics vary along the axial direction, Z:

$$B_r(r,\theta,z) = \sum_{n=1}^{\infty}\left[B_n(z) + \sum_{l=1}^{\infty}\frac{(-1)^l (n-1)!(2l+n)}{2^{2l}\, l!(l+n)!} B_n^{[2l]}\, r^{2l}\right]\left(\frac{r}{R_{ref}}\right)^{n-1}\sin(n\theta)$$

$$+ \sum_{n=1}^{\infty}\left[A_n(z) + \sum_{l=1}^{\infty}\frac{(-1)^l (n-1)!(2l+n)}{2^{2l}\, l!(l+n)!} A_n^{[2l]}\, r^{2l}\right]\left(\frac{r}{R_{ref}}\right)^{n-1}\cos(n\theta)$$

$$B_\theta(r,\theta,z) = \sum_{n=1}^{\infty}\left[B_n(z) + \sum_{l=1}^{\infty}\frac{(-1)^l n!}{2^{2l}\, l!(l+n)!} B_n^{[2l]}\, r^{2l}\right]\left(\frac{r}{R_{ref}}\right)^{n-1}\cos(n\theta)$$

$$- \sum_{n=1}^{\infty}\left[A_n(z) + \sum_{l=1}^{\infty}\frac{(-1)^l n!}{2^{2l}\, l!(l+n)!} A_n^{[2l]}\, r^{2l}\right]\left(\frac{r}{R_{ref}}\right)^{n-1}\sin(n\theta)$$

where the index [2ℓ] denotes $(2\ell)^{th}$ derivative with respect to *z*. If integral values are considered between Z_1 and Z_2 such that all derivatives are zero at the ends, then the above expression reduces to the 2-D expansion.

Interpretation of Harmonics

$$B_{n+1}(\text{European}) = B_n(\text{US}) = \frac{R_{ref}^n}{n!} \left(\frac{\partial^n B_y}{\partial x^n} \right) \Bigg|_{x=0;y=0}$$

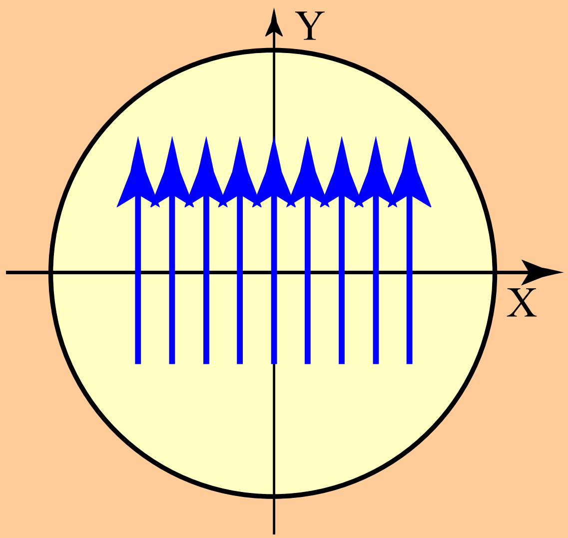
$$A_{n+1}(\text{European}) = A_n(\text{US}) = \frac{R_{ref}^n}{n!} \left(\frac{\partial^n B_x}{\partial x^n} \right) \Bigg|_{x=0;y=0} \qquad n \geq 0$$

$B_y = \text{Constant} \Rightarrow$ Dipole Only

$(dB_y/dx) = \text{Constant} \Rightarrow$ Dipole plus Quadrupole

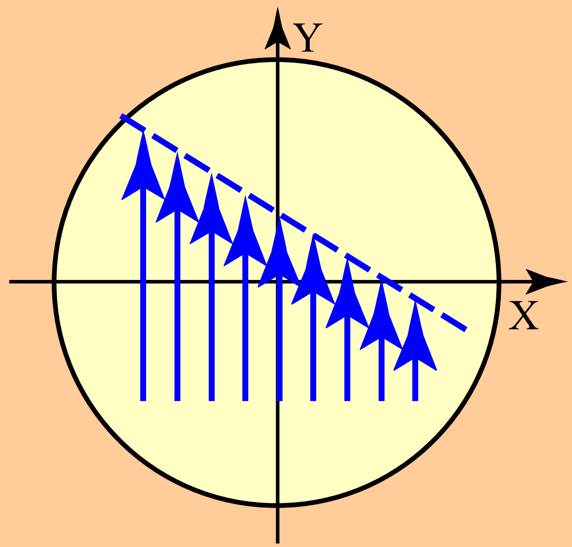
and so on ...

Examples of Harmonics



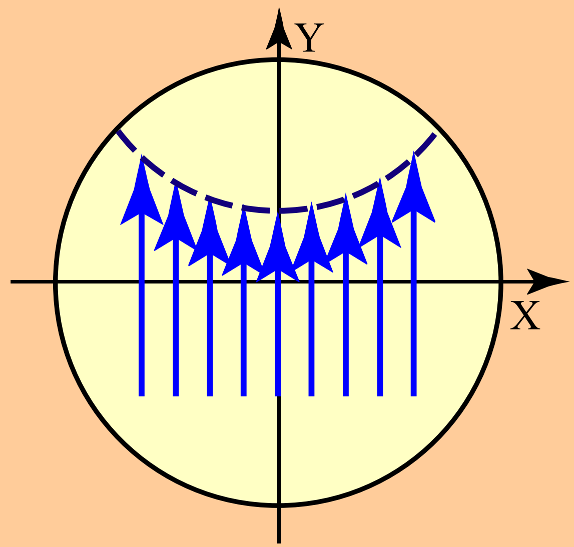
$B_y = B_0$ (Constant)

Normal Dipole



$B_y = B_0 + G.x$

Normal Dipole
+
Normal Quadrupole



$B_y = B_0 + A.x^2$

Normal Dipole
+
Normal Sextupole

Normalized Coefficients: Multipoles

- The coefficients B_n and A_n denote the absolute strength of the *n*-th harmonic, and are thus a function of the magnet excitation.
- The variation in the *shape* of the field as a function of excitation is best described using coefficients normalized by a suitable ***reference field***, often chosen to be the amplitude of the most dominant term in the harmonic expansion. The normalized coefficients are also referred to as ***multipoles***.

Normalized Coefficients: Multipoles

$$B_y + iB_x = \sum_{n=n_0}^{\infty} [B_n + iA_n] \left(\frac{x+iy}{R_{ref}} \right)^{n-n_0} \quad ;$$

$n_0 = 0$: US
$n_0 = 1$: European

$$= B_{ref} \sum_{n=n_0}^{\infty} [b_n + ia_n] \left(\frac{x+iy}{R_{ref}} \right)^{n-n_0} \quad ; \text{where}$$

$$b_n = B_n / B_{ref}; \qquad a_n = A_n / B_{ref}$$

For a 2m-pole magnet, B_{ref} chosen as $|B_m + iA_m|$

(b_n, a_n) independent of current: LINEAR SYSTEM

$(b_n, a_n) \times 10^4$ = Normal & Skew Multipoles in "UNITS"

Properties of Harmonics

- The Normal and Skew harmonics represent coefficients of expansion in a power series for the field components.
- The harmonics allow computation of field everywhere in the aperture (within a circle of convergence) using only a few numbers.
- These coefficients obviously depend on the choice of origin and orientation of the coordinate system. Measured harmonics, therefore, often need to be ***centered*** and ***rotated***.

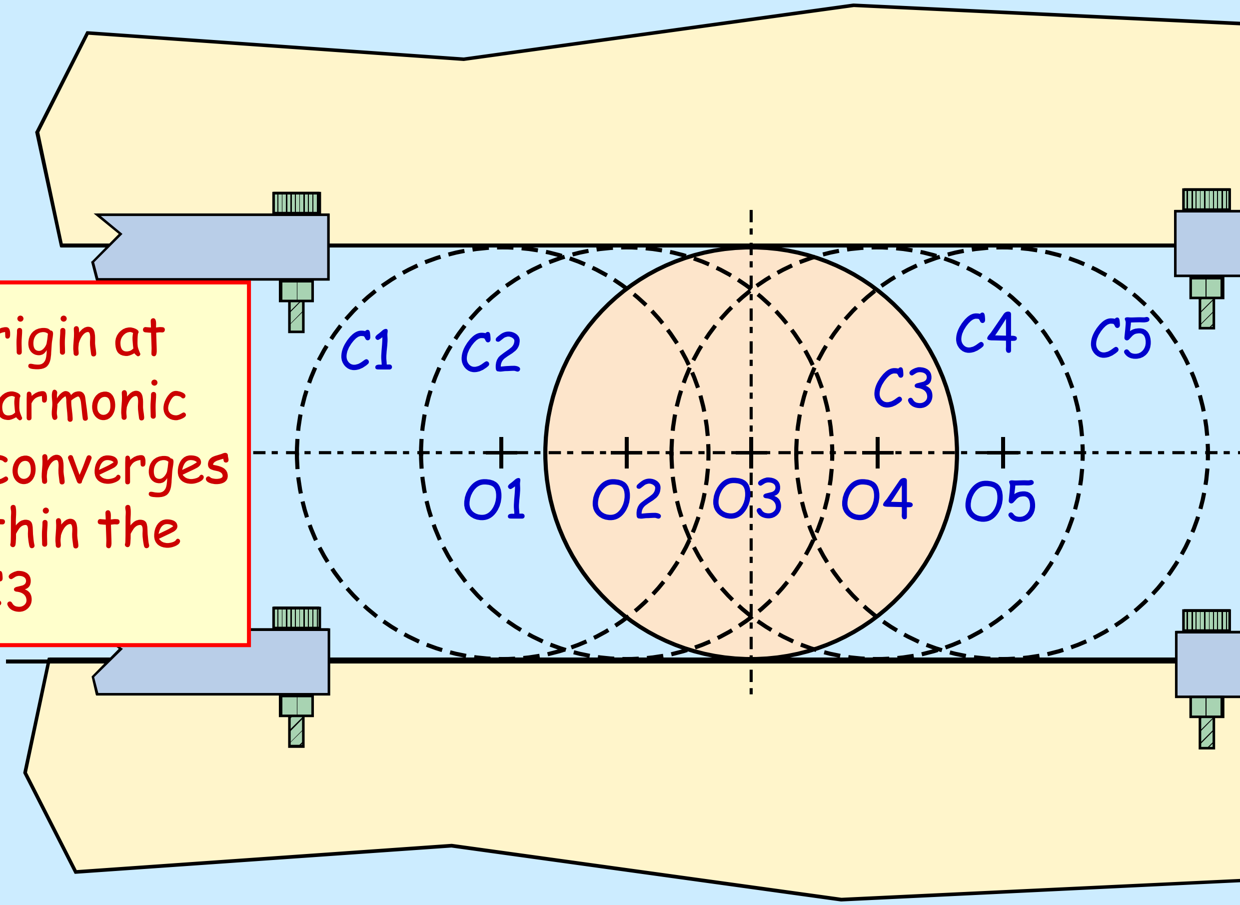
Field in a Non-circular Aperture

The 2-D field expansion in a harmonic series is valid only within the circle of convergence, which extends from the origin to the nearest current element or a magnetic material ("singularity").

For non-circular apertures, a single series expansion may not cover the entire "source-free region", even though the complex field $B_y + iB_x$ is an analytic function of $(x + iy)$ throughout the aperture. One can circumvent the problem by defining several series expansions, each centered at a different origin.

Field in a Non-circular Aperture

With origin at O3, a harmonic series converges only within the circle C3



With the origin shifted to O1, O2, O4, O5, a NEW harmonic series is valid within circles C1, C2, C4, C5.

By having a significant overlap between the various circles of convergence, one can verify the integrity and accuracy of data by comparing results in the overlap regions.

Calculation of Field Outside Convergence Radius

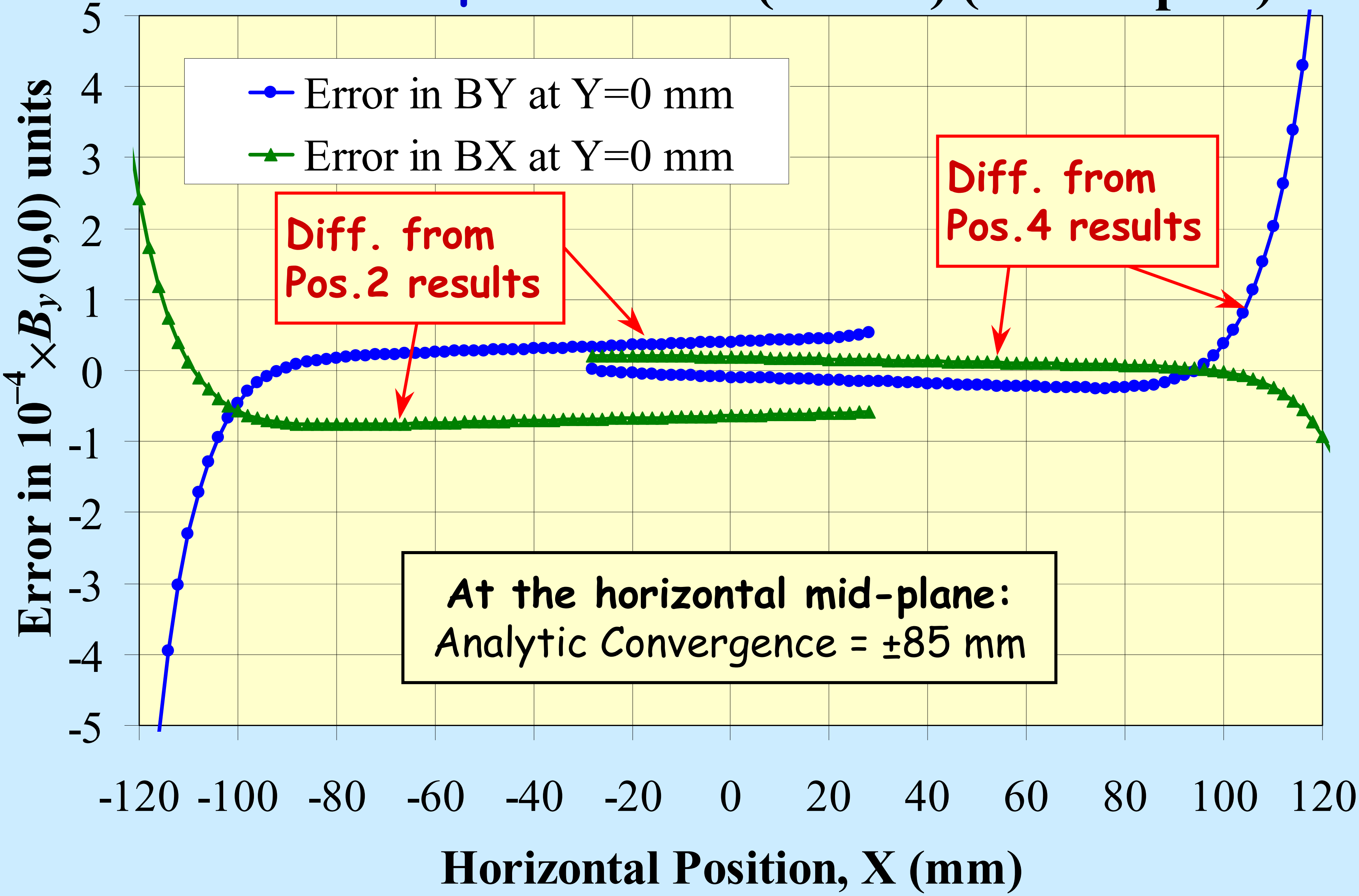
- The field at any point (x, y) can be computed from the normal and skew harmonics measured at (x_0, y_0) using:

$$B_y(x,y) + iB_x(x,y) = \sum_{n=0}^{\infty} \left(B_n + iA_n\right) \left(\frac{(x - x_0) + i(y - y_0)}{R_{ref}} \right)^n$$

- For accurate calculations of field, the origin (x_0, y_0) should be chosen such that the field point (x, y) is as close as possible, and is well within the circle of convergence defined by the pole tips.
- In special situations, such as when the higher order terms are identically zero, convergence of the series may not be a problem, and it ***may*** be possible to calculate the field outside the radius of convergence using the same (B_n, A_n).

Error Due to Extending Radius of Pos.3

Practical Example: SD1717 (1 GeV) (SNS Dipole)



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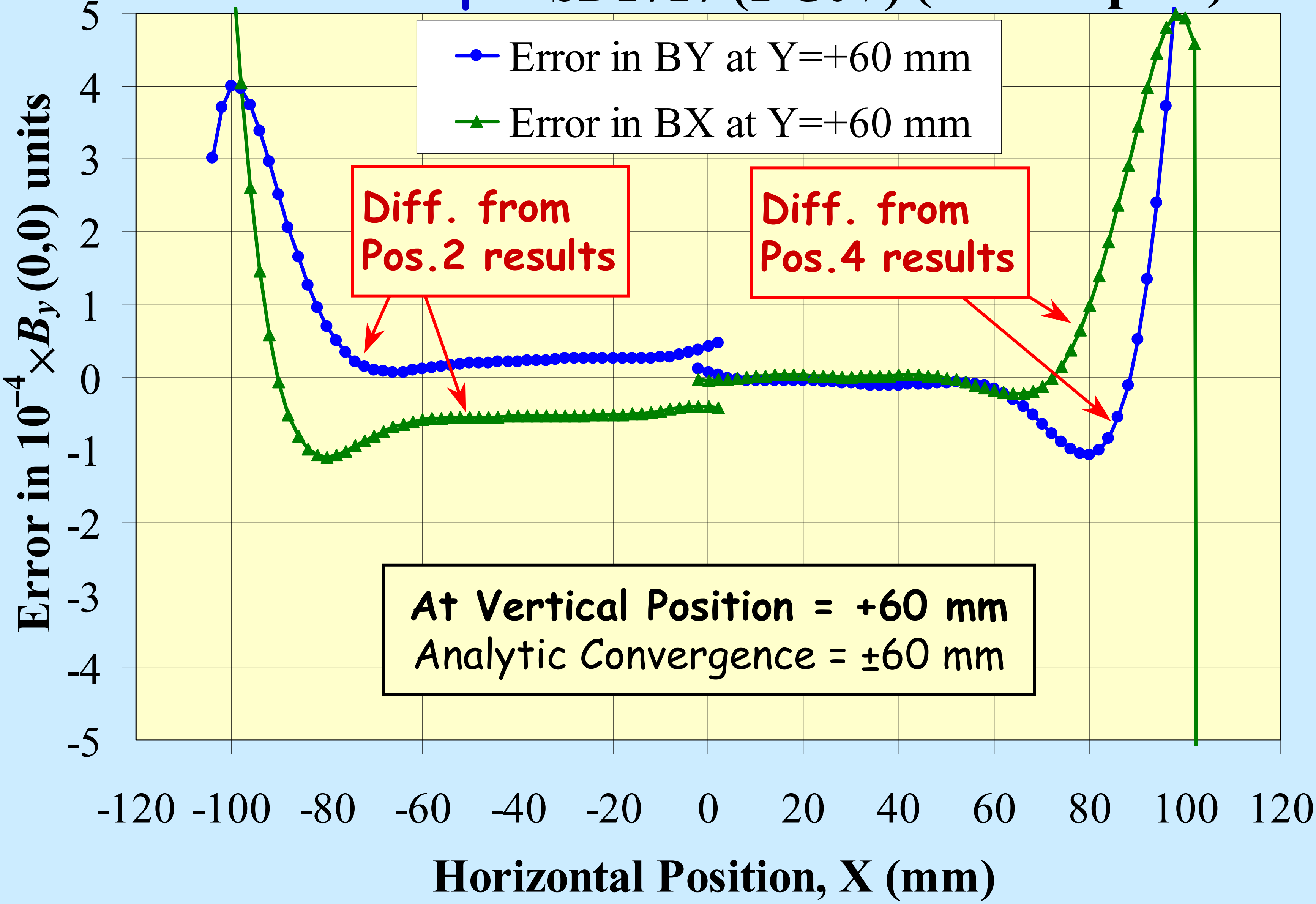

The Vector Potential

Scalar potential approach does not provide a relationship between the currents and the field.

From Maxwell's equations:

$$\nabla . \mathbf{B} = 0; \quad \therefore \mathbf{B} = \nabla \times \mathbf{A}$$

A is called the Vector Potential

$$\nabla \times \mathbf{B} = \mu_0 (\nabla \times \mathbf{H}) = \mu_0 \mathbf{J}$$

In "free space", $\mathbf{B} = \mu_0 \mathbf{H}$

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = \mu_0 (\nabla \times \mathbf{H}) = \mu_0 \mathbf{J}$$

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$$

Poisson's Equation

$$\mathbf{A}(\mathbf{r}) = \left(\frac{\mu_0}{4\pi}\right) \int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}'$$

Summary

- The 2-D field , away from the ends, in the aperture of a typical accelerator magnet, can be described by a simple power series, valid within a circle extending to the nearest current source or magnetic material.
- A similar 2-D expansion is also valid for 3-D fields if one considers only integrated values of the field components such that there is no axial variation at the boundaries of the integration interval.

Summary (Contd.)

- The expansion coefficients may be interpreted as spatial derivatives of the field components.
- The expansion coefficients, or harmonics, depend on the choice of coordinate frame. This demands a careful description of the frame when quoting results of measurements. Similarly, users of the data also need to pay close attention to the coordinate definition.

Summary (Contd.)

- The complex field, $\boldsymbol{B}(\boldsymbol{z}) = B_y + iB_x$, is an analytic function of the complex variable, $\boldsymbol{z}$.
- For non-circular apertures, one can describe the field in the entire aperture by defining several series expansions centered at different points in the aperture (***analytic continuation***).
- Scalar potential approach is unsuitable for establishing a relationship between the current and the field. A vector potential approach is more general.