Magnetic Fields Produced by Current Distributions

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The Complex Potential in 2-D

The complex field, $\boldsymbol{B}(\boldsymbol{z}) = B_y(x, y) + i\, B_x(x, y)$ is given by:

$$\boldsymbol{B}(\boldsymbol{z}) = B_y(x,y) + iB_x(x,y) = \sum_{n=1}^{\infty}\left[C(n)\exp(-in\alpha_n)\right]\left(\frac{\boldsymbol{z}}{R_{ref}}\right)^{n-1}$$

which is an analytic function of the complex variable $\boldsymbol{z} = x + iy$. Accordingly, we define a **Complex Potential** $\boldsymbol{W}(\boldsymbol{z})$ such that:

$$\boxed{\boldsymbol{B}(\boldsymbol{z}) = -\frac{d\boldsymbol{W}(\boldsymbol{z})}{d\boldsymbol{z}}}$$

It can be shown that the real and imaginary parts of this complex potential are nothing but the vector and the scalar potentials respectively.

Complex, Vector & Scalar Potentials

$$W(z) = W_r(x,y) + iW_i(x,y) \quad \text{(Complex Potential)}$$

$$\frac{dW(z)}{dx} = \frac{dW(z)}{dz} \cdot \frac{dz}{dx} = -(B_y + iB_x).1 = \left(\frac{\partial W_r}{\partial x}\right) + i\left(\frac{\partial W_i}{\partial x}\right)$$

$$\therefore \left(\frac{\partial W_r}{\partial x}\right) = -B_y = \left(\frac{\partial A_z}{\partial x}\right); \quad \left(\frac{\partial W_i}{\partial x}\right) = -B_x = \left(\frac{\partial \Phi_m}{\partial x}\right)$$

$$\frac{dW(z)}{dy} = \frac{dW(z)}{dz} \cdot \frac{dz}{dy} = -(B_y + iB_x).i = \left(\frac{\partial W_r}{\partial y}\right) + i\left(\frac{\partial W_i}{\partial y}\right)$$

$$\therefore \left(\frac{\partial W_r}{\partial y}\right) = B_x = \left(\frac{\partial A_z}{\partial y}\right); \quad \left(\frac{\partial W_i}{\partial y}\right) = -B_y = \left(\frac{\partial \Phi_m}{\partial y}\right)$$

$$\mathrm{Re}[W(z)] = A_z \text{ (Vector Potential)}$$

$$\mathrm{Im}[W(z)] = \Phi_m \text{ (Scalar Potential)}$$

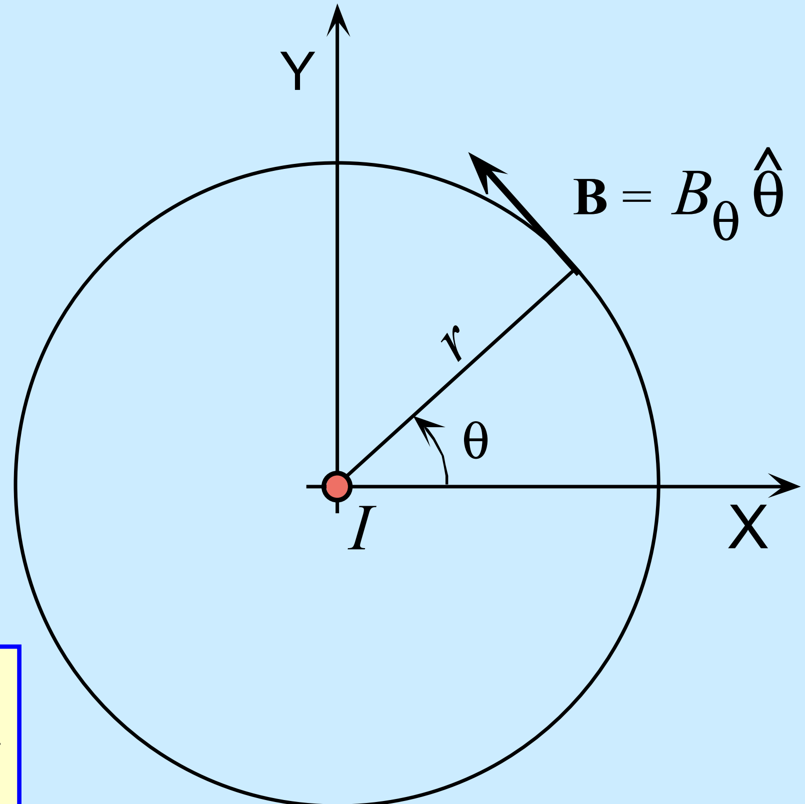
Current Filament at Origin

From Ampère's law:

$$\mathbf{B} = \mu_0 \mathbf{H} = \frac{\mu_0 I}{2\pi r}\hat{\theta}$$

$$B_x = B_r \cos\theta - B_\theta \sin\theta = -\frac{\mu_0 I}{2\pi r}\sin\theta$$

$$B_y = B_r \sin\theta + B_\theta \cos\theta = \frac{\mu_0 I}{2\pi r}\cos\theta$$



$$\boldsymbol{B}(z) = B_y + iB_x = \frac{\mu_0 I}{2\pi r.\exp(i\theta)} = \frac{\mu_0 I}{2\pi z}$$

$$\boldsymbol{W}(z) = -\int B(z)dz = -\left(\frac{\mu_0 I}{2\pi}\right)\ln(z) + \text{constant}$$

$$= -\left(\frac{\mu_0 I}{2\pi}\right)[\ln(r) + i\theta] + \text{constant} \;=\; A_z + i\Phi_m$$

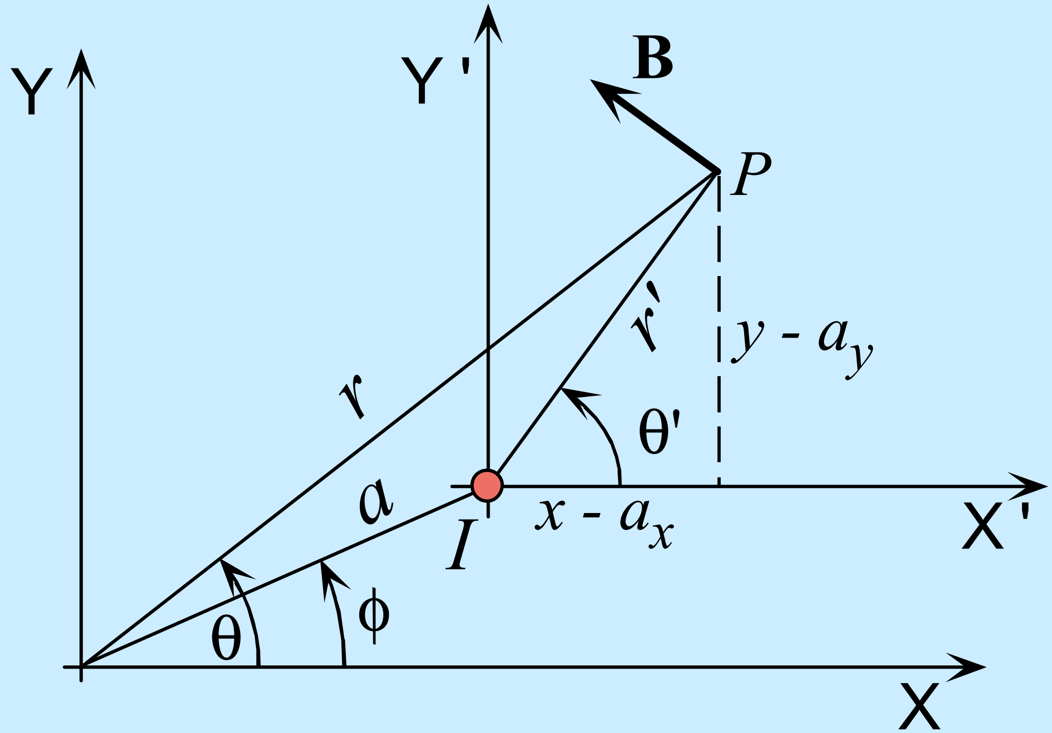
Current Filament at Arbitrary Point

Current filament located at

$$\boldsymbol{a} = a\exp(i\phi) = a_x + ia_y$$

$$\boldsymbol{B}(\boldsymbol{z'}) = B_{y'} + iB_{x'} = \frac{\mu_0 I}{2\pi \boldsymbol{z'}}$$

$$\boldsymbol{B}(\boldsymbol{z}) = B_y + iB_x = B_{y'} + iB_{x'} = \left(\frac{\mu_o I}{2\pi \boldsymbol{z'}}\right) = \left(\frac{\mu_o I}{2\pi(\boldsymbol{z} - \boldsymbol{a})}\right)$$



$$W(\boldsymbol{z}) = -\int \boldsymbol{B}(\boldsymbol{z})d\boldsymbol{z} = -\left(\frac{\mu_0 I}{2\pi}\right)\ln(\boldsymbol{z} - \boldsymbol{a}) + \text{const.}$$

$$= -\left(\frac{\mu_0 I}{2\pi}\right)[\ln(r') + i\theta'] + \text{const.} = A_z + i\Phi_m$$

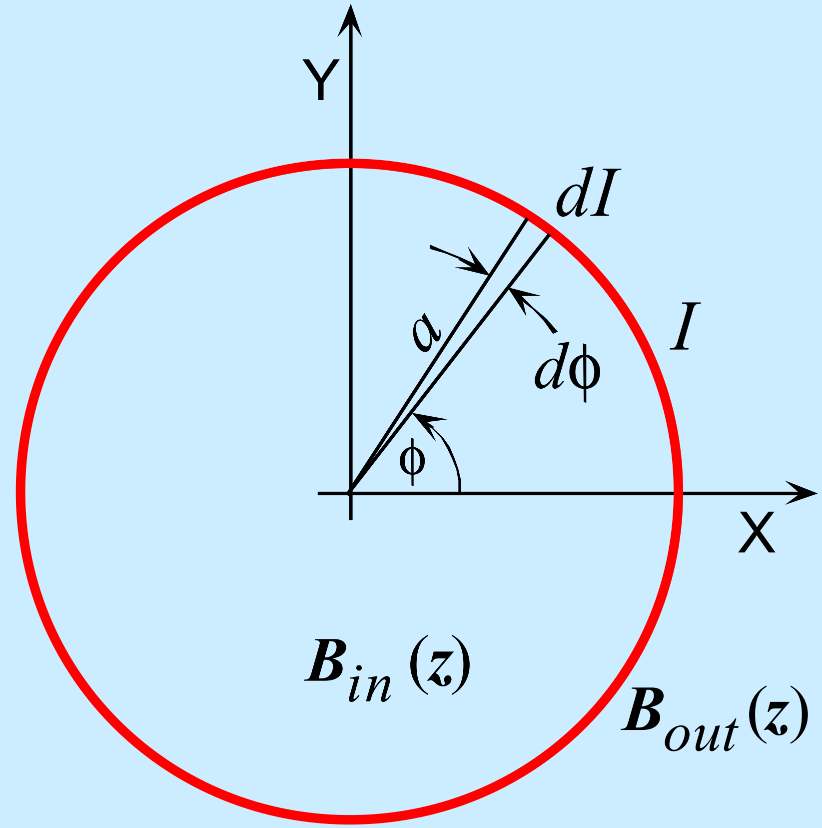
Uniform Cylindrical Current Shell

$$\text{Choose}: z' = a\exp(i\phi)$$

$$dI = \left(\frac{I}{2\pi}\right)d\phi = \left(\frac{I}{2\pi i}\right)\left(\frac{dz'}{z'}\right)$$

$$d\boldsymbol{B}(z) = \left(\frac{\mu_0}{2\pi}\right)\left(\frac{I}{2\pi i}\right)\left[\frac{dz'}{z'(z-z')}\right]$$

$$\boldsymbol{B}(z) = \left(\frac{\mu_0 I}{4\pi^2 z i}\right)\left[\oint\frac{dz'}{z'} - \oint\frac{dz'}{z'-z}\right]$$



$$\boxed{\boldsymbol{B}_{in}(z) = 0 \quad (|z| < a)}$$

Field is zero at points inside the shell.

$$\boxed{\boldsymbol{B}_{out}(z) = \left(\frac{\mu_0 I}{2\pi z}\right)}$$

For points outside, the current shell behaves as a filament located at the center of the shell.

Uniform Solid Cylindrical Conductor

Take a cylindrical shell element of radius ξ

$$dI = J.2\pi\xi d\xi; \text{ where } J = \frac{I}{\pi a^2}$$

For any point, *P*, inside the cylinder:

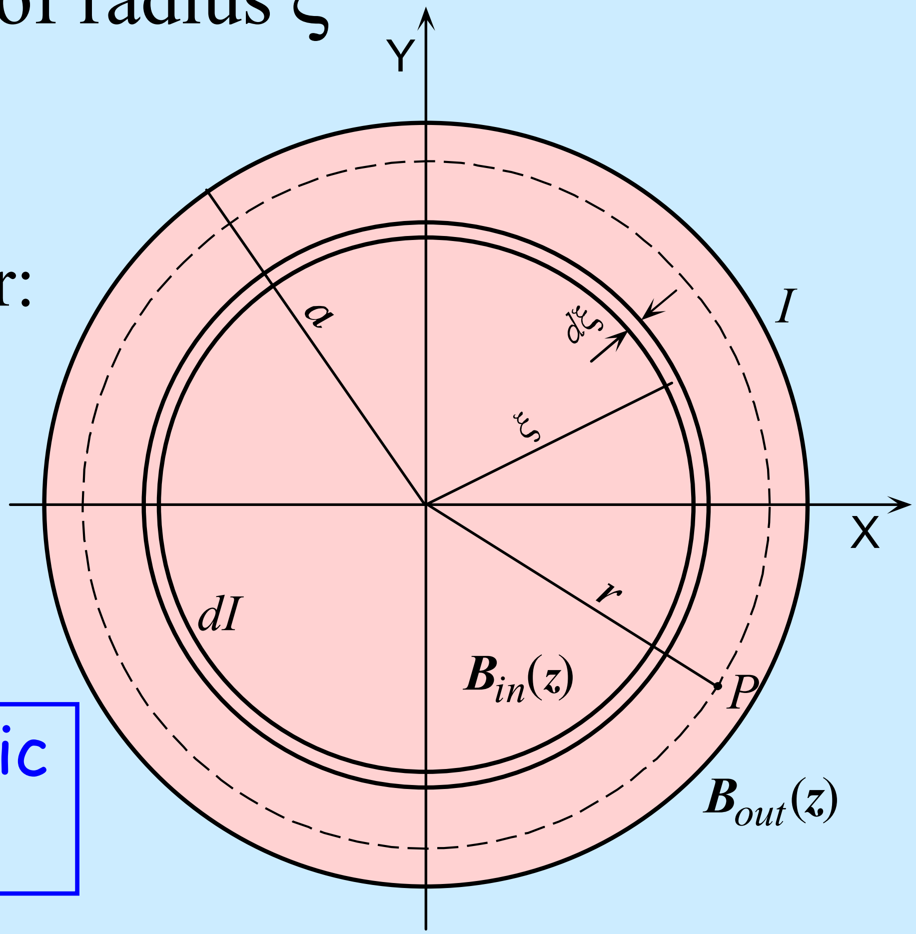
$$\boldsymbol{B}_{in}(\boldsymbol{z}) = \left(\frac{\mu_o J}{\boldsymbol{z}}\right) \cdot \int_0^r \xi . d\xi = \left(\frac{\mu_o J r^2}{2\boldsymbol{z}}\right)$$

$$\boldsymbol{B}_{in}(\boldsymbol{z}) = \left(\frac{\mu_o J}{2}\right)\boldsymbol{z}^*$$

Not an analytic function of *z*

$$\boldsymbol{B}_{out}(\boldsymbol{z}) = \left(\frac{\mu_0 I}{2\pi \boldsymbol{z}}\right) = \left(\frac{\mu_0 J}{2}\right)\left(\frac{a^2}{\boldsymbol{z}}\right)$$

Analytic function of *z* for points outside the cylinder



Overlapping Cylinders: Pure Dipole

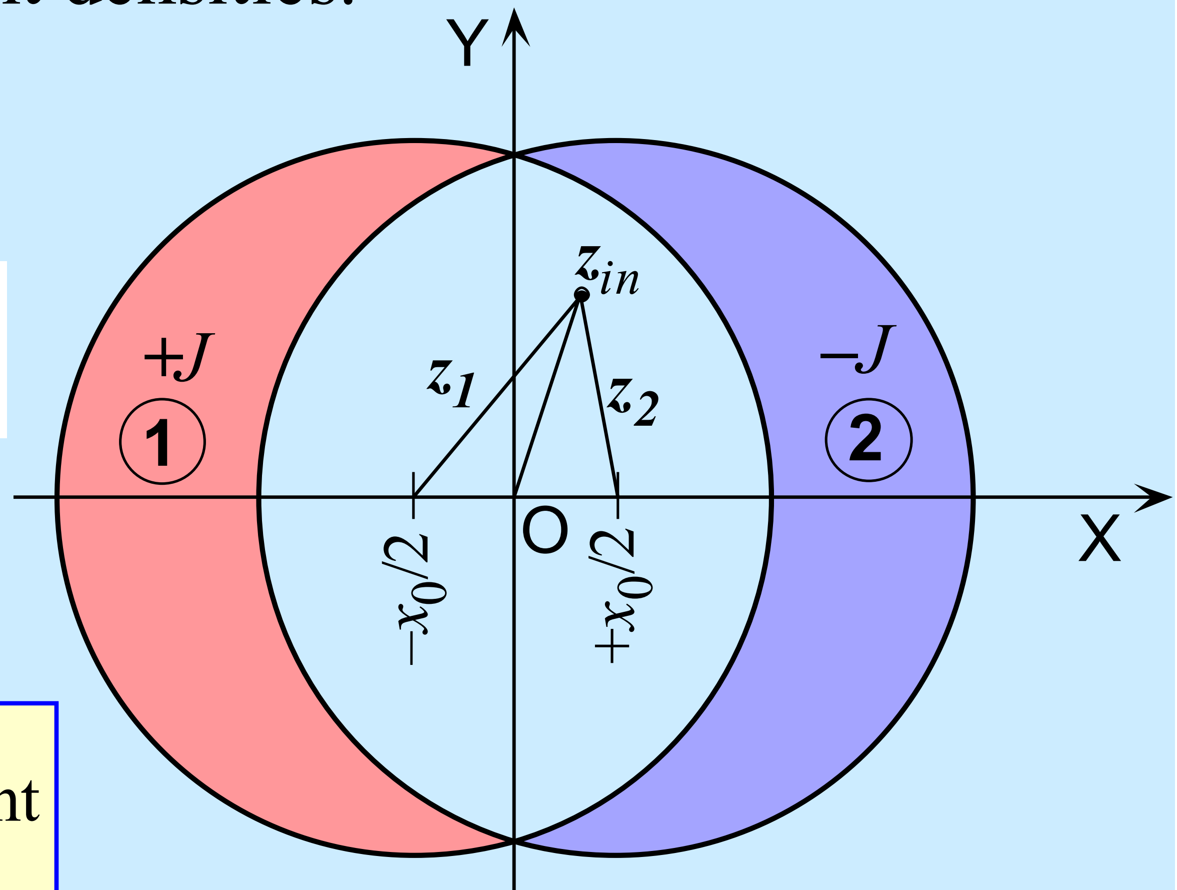
Consider two overlapping solid cylinders carrying equal and opposite current densities.

For any point, z_{in}, inside the current free region:

$$z_1 = z_{in} + \frac{x_0}{2}; \quad z_2 = z_{in} - \frac{x_0}{2}$$

$$\boldsymbol{B}(z_{in}) = \left(\frac{\mu_0 J}{2}\right) \cdot \left(z_1^* - z_2^*\right)$$

$$\boxed{\boldsymbol{B}(z_{in}) = \left(\frac{\mu_0 J x_0}{2}\right) = \text{Constant}}$$



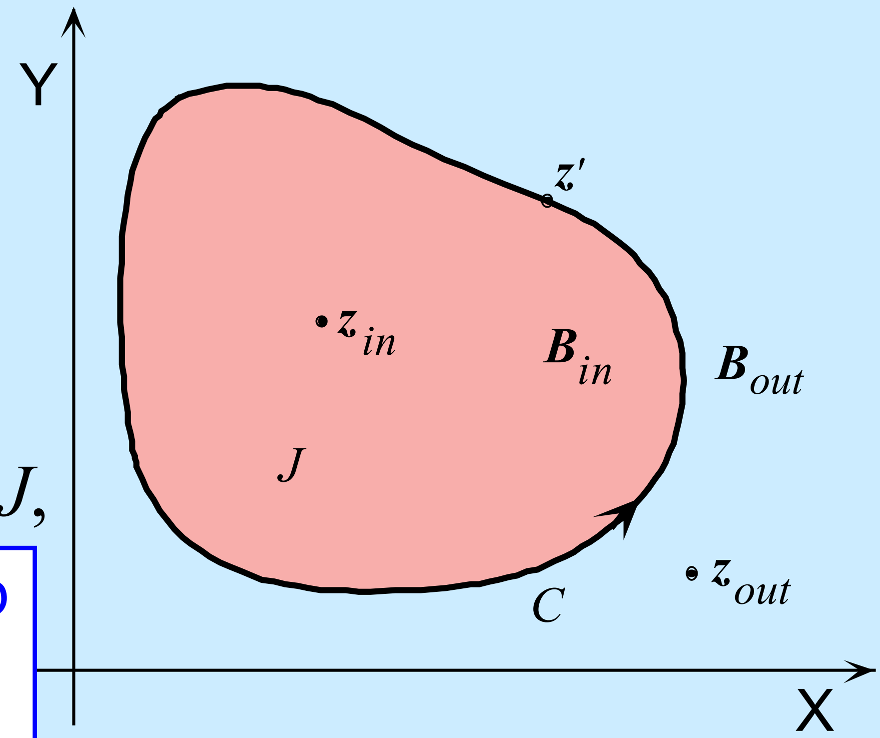
This represents a pure Dipole Field in the "aperture"

Conductor of Arbitrary Cross Section

At any point inside the conductor:

$$(\nabla \times \boldsymbol{B})_z = \left(\frac{\partial B_y}{\partial x}\right) - \left(\frac{\partial B_x}{\partial y}\right) = \mu_0 J_z(x, y)$$

$$\nabla \cdot \boldsymbol{B} = \left(\frac{\partial Bx}{\partial x}\right) + \left(\frac{\partial B_y}{\partial y}\right) = 0$$



For constant current density, $J_z(x,y)=J$,

$$\boldsymbol{F}(z) = \boldsymbol{B}(z) - \left(\frac{\mu_0 J}{2}\right) z^*$$

Can be shown to be an analytic function of z

$$\boldsymbol{B}(z) = \left(\frac{\mu_0 J}{4\pi}\right) i \oint \frac{(z' - z)^* \, dz'}{z' - z}$$

Integral Formula for Field at any point

This formula allows computation of the field by integrating along only the *boundary* of the conductor, instead of the entire volume

Derivation of Integral Formula

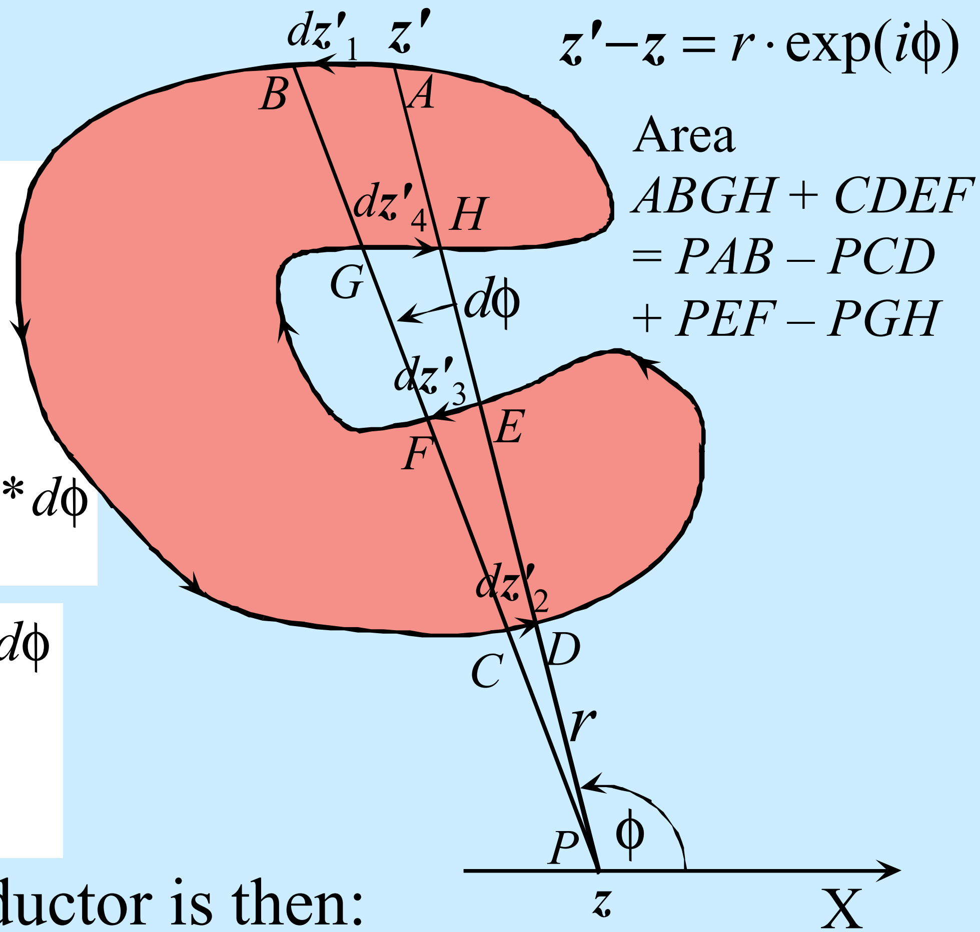
Field at point P due to a triangular wedge PAB is given by:

$$dB(z) = -\left(\frac{\mu_0 J\, d\phi}{2\pi}\right)\int_0^r \frac{\rho\, d\rho}{\rho \exp(i\phi)}$$

$$= -\left(\frac{\mu_0 J}{2\pi}\right) r \exp(-i\phi)\, d\phi = -\left(\frac{\mu_0 J}{2\pi}\right)(z' - z)^*\, d\phi$$

$$z' - z = r \exp(i\phi); \quad dz'/(z' - z) = (dr/r) + i\, d\phi$$

$$\therefore d\phi = \left(\frac{1}{2i}\right)\left(\frac{dz'}{z' - z} - \frac{dz'^*}{z'^* - z^*}\right)$$

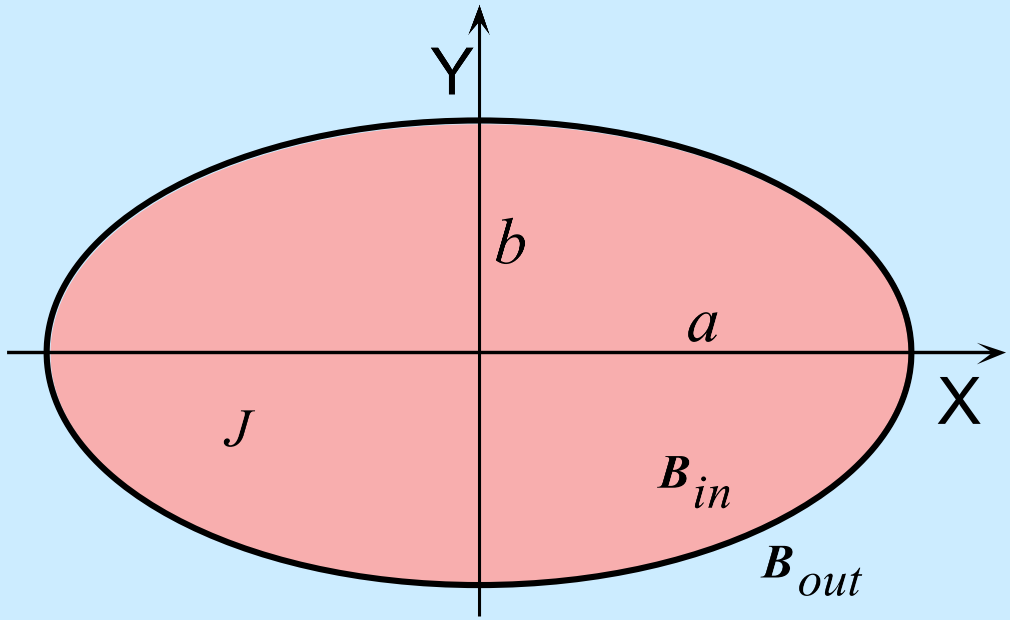


The total field due to the entire conductor is then:

$$B(z) = \left(\frac{\mu_0 J}{4\pi}\right) i \left[\oint \frac{(z' - z)^*\, dz'}{z' - z} - \oint dz'^*\right] = \left(\frac{\mu_0 J}{4\pi}\right) i \oint \frac{(z' - z)^*\, dz'}{z' - z}$$

Reference: R.A. Beth, *J. Appl. Phys.* **40**(12), 4782-6 (1969)

Solid Elliptical Cross Section Conductor



The integral formula gives in this case:

$$\boldsymbol{B}_{in}(\boldsymbol{z}) = \frac{\mu_o J}{(a+b)}[bx - iay]$$

$$\boldsymbol{B}_{out}(\boldsymbol{z}) = \left(\frac{\mu_o J}{2}\right)\left[\frac{2ab}{\boldsymbol{z} + \sqrt{\boldsymbol{z}^2 - (a^2 - b^2)}}\right]$$

For $b = a$, these reduce to the expressions for a circular cross section.

Reference: R.A. Beth, *J. Appl. Phys.* **38**(12), 4689-92 (1967)

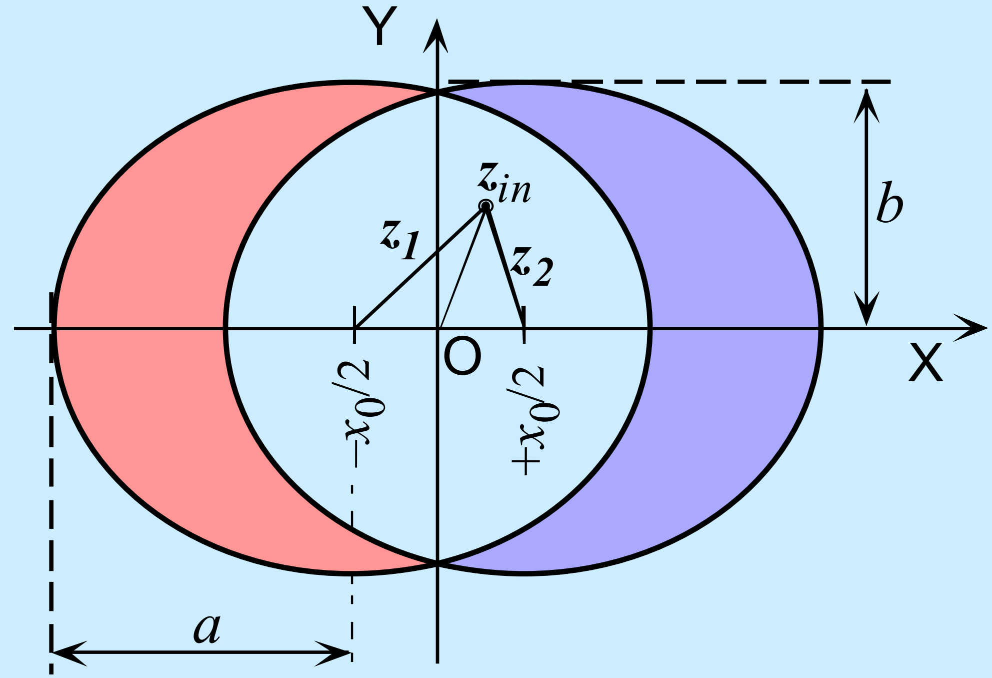
Overlapping Ellipses: Pure Dipole

Consider two overlapping ellipses carrying equal and opposite current densities.

For any point, z_{in}, inside the current free region:

$$\boldsymbol{B}(z_{in}) = \frac{\mu_o J}{(a+b)}\left[b\left(x + \frac{x_0}{2}\right) - iay - b\left(x - \frac{x_0}{2}\right) + iay \right]$$

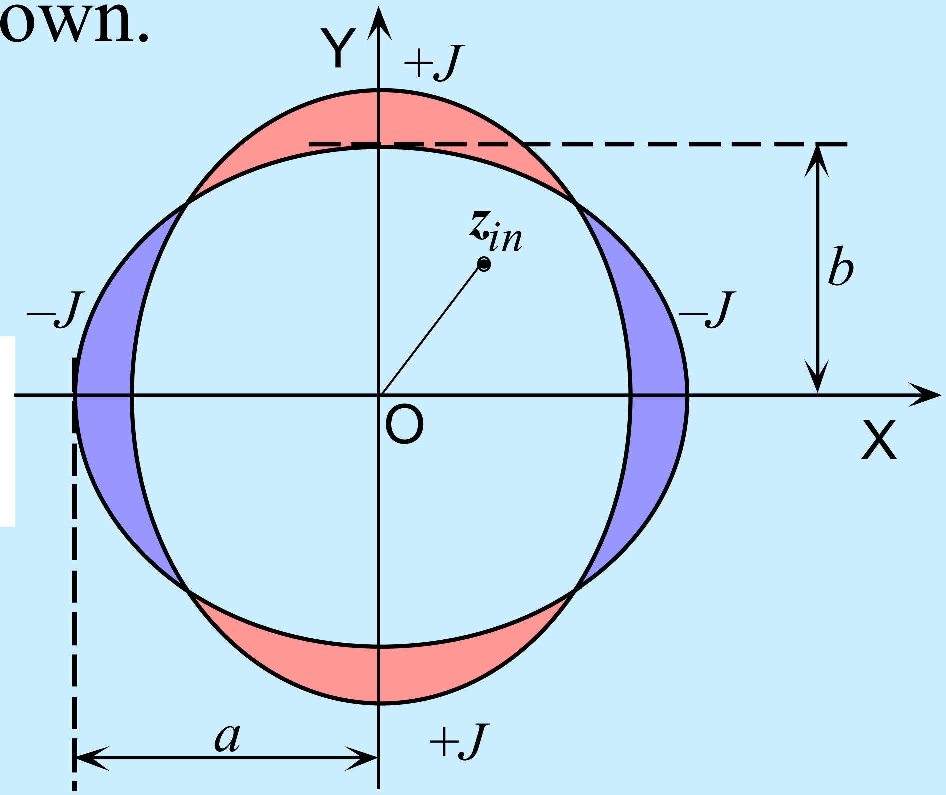
$$\boxed{\boldsymbol{B}(z_{in}) = \frac{\mu_o J b x_0}{(a+b)} = \text{constant}}$$



This represents a pure Dipole Field in the "aperture"

Overlapping Ellipses: Pure Quadrupole

Consider two overlapping ellipses carrying equal and opposite current densities, as shown.



For any point, $\boldsymbol{z}_{in}$, inside the current free region:

$$\boldsymbol{B}(\boldsymbol{z}_{in}) = \frac{\mu_o J}{(a+b)}\left[-bx + iay + ax - iby\right]$$

$$\boxed{\boldsymbol{B}(\boldsymbol{z}_{in}) = \frac{\mu_o J(a-b)}{(a+b)}(x+iy)}$$

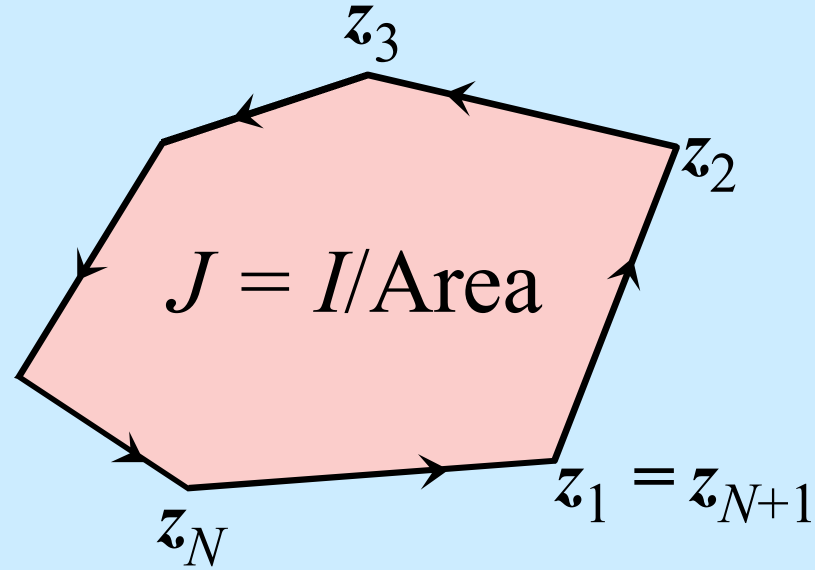
$$\boxed{B_y(\boldsymbol{z}) = \frac{\mu_o J(a-b)}{(a+b)}x\,; \quad B_x(\boldsymbol{z}) = \frac{\mu_o J(a-b)}{(a+b)}y}$$

This represents a pure Quadrupole Field in the "aperture"

Conductor of Polygonal Cross Section

$$\boldsymbol{B}(z) = \left(\frac{\mu_0 J}{4\pi}\right) i \oint \frac{z'^* - z^*}{z' - z}\, dz'$$

$$= \left(\frac{\mu_0 J}{4\pi}\right) \sum_{j=1}^{N} I_j(z)$$



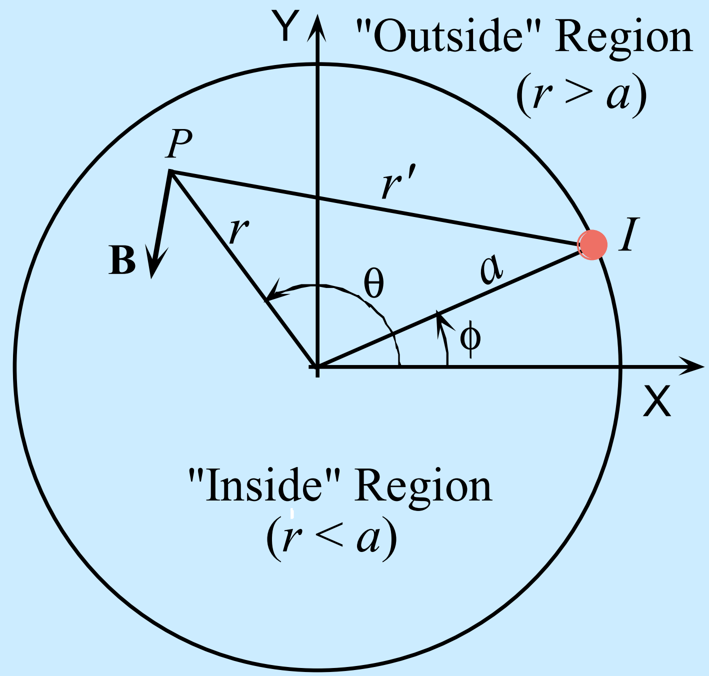
$$I_j(z) = i\left[\left(z_j^* - z^*\right) + \left(\frac{z_{j+1}^* - z_j^*}{z_{j+1} - z_j}\right)\left(z - z_j\right)\right] \ln\left(\frac{z_{j+1} - z}{z_j - z}\right); \quad z \neq z_j \text{ and } z \neq z_{j+1}$$

Use $I_j(z) = 0$ if $z = z_j$; or if $z = z_{j+1}$

Cross sectional area, A, is given by: $$A = \frac{1}{2} \sum_{j=1}^{N} \left(x_j\, y_{j+1} - x_{j+1}\, y_j\right)$$

Harmonic Expansion:Current Filament

For any point, P, inside circle of radius a:



$$\boldsymbol{B}_{in}(z) = \left(\frac{\mu_0 I}{2\pi}\right).(\boldsymbol{z} - \boldsymbol{a})^{-1}$$

$$= -\left(\frac{\mu_0 I}{2\pi a \exp(i\phi)}\right)\left[1 - \left(\frac{r}{a}\right)\exp\{i(\theta - \phi)\}\right]^{-1}$$

$$(1-\xi)^{-1} = \sum_{n=1}^{\infty} \xi^{n-1}$$

Binomial Expansion

$$\boxed{\boldsymbol{B}_{in}(z) = -\left(\frac{\mu_0 I}{2\pi a}\right)\sum_{n=1}^{\infty}\exp(-in\phi)\left(\frac{R_{ref}}{a}\right)^{n-1}\left(\frac{\boldsymbol{z}}{R_{ref}}\right)^{n-1} \equiv \sum_{n=1}^{\infty}(B_n + iA_n)\left(\frac{\boldsymbol{z}}{R_{ref}}\right)^{n-1}}$$

$$\boxed{B_n = -\left(\frac{\mu_0 I}{2\pi a}\right).\left(\frac{R_{ref}}{a}\right)^{n-1}\cos(n\phi); \quad A_n = \left(\frac{\mu_0 I}{2\pi a}\right).\left(\frac{R_{ref}}{a}\right)^{n-1}\sin(n\phi)}$$

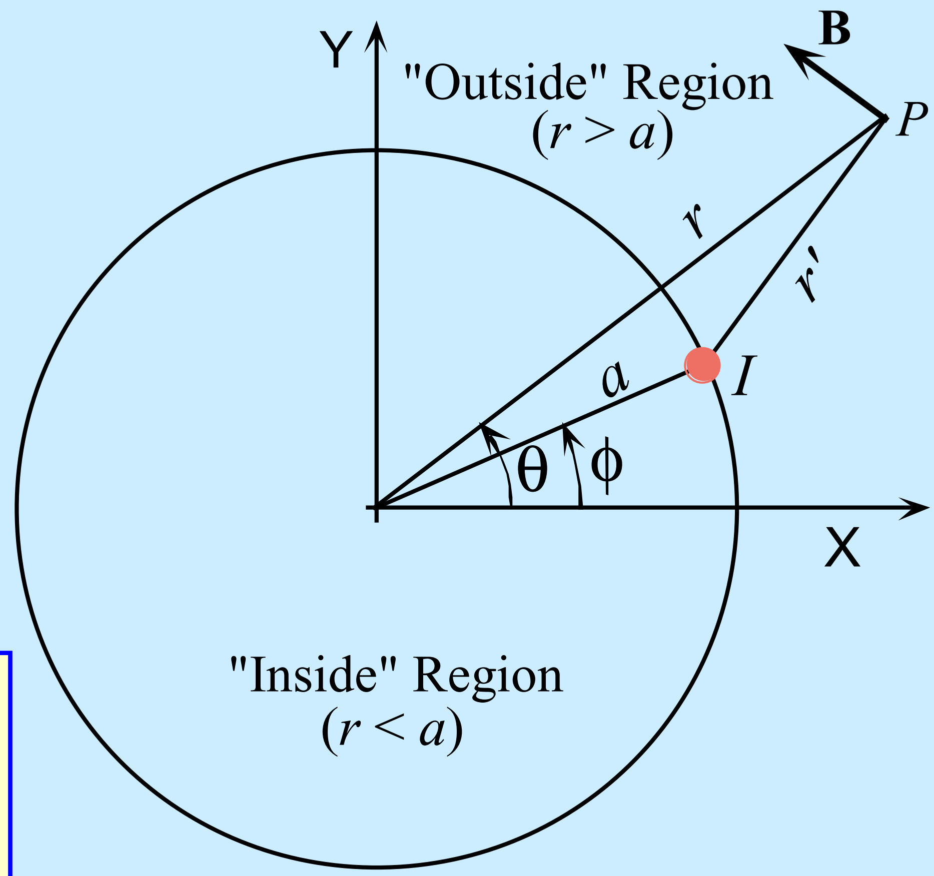
n = 1 is Dipole

Series Expansion for Outside Region

For any point, P, outside circle of radius a:

$$\boldsymbol{B}_{out}(\boldsymbol{z}) = \left(\frac{\mu_0 I}{2\pi}\right).(\boldsymbol{z}-\boldsymbol{a})^{-1}$$

$$= \left(\frac{\mu_0 I}{2\pi r \exp(i\theta)}\right)\left[1-\left(\frac{a}{r}\right)\exp\{i(\phi-\theta)\}\right]^{-1}$$



$$(1-\xi)^{-1} = 1+\sum_{n=1}^{\infty}\xi^n$$

Binomial Expansion

$$\boxed{\boldsymbol{B}_{out}(\boldsymbol{z}) = \left(\frac{\mu_0 I}{2\pi \boldsymbol{z}}\right)\left[1+\sum_{n=1}^{\infty}\exp(in\phi)\left(\frac{a}{\boldsymbol{z}}\right)^n\right]}$$

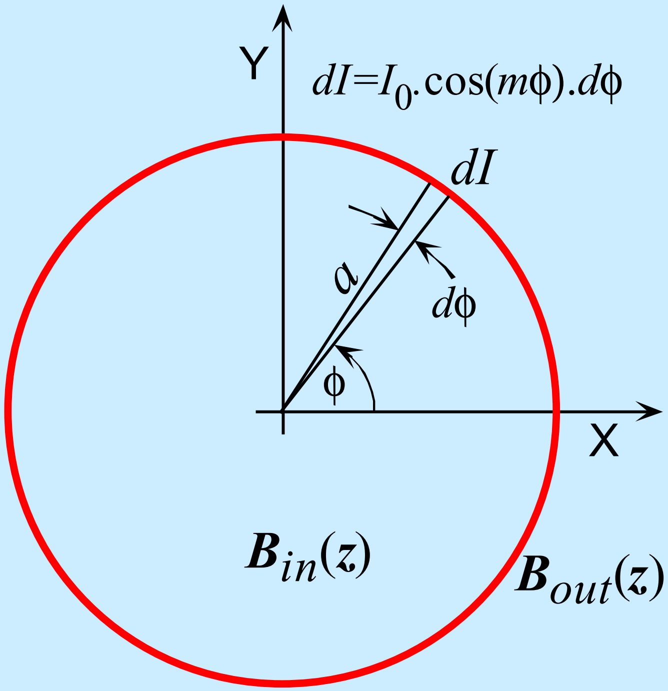
$$\boldsymbol{B}_{out}(\boldsymbol{z}) \xrightarrow[|\boldsymbol{z}|\to\infty]{} \left(\frac{\mu_0 I}{2\pi \boldsymbol{z}}\right)$$

(As expected for a current filament)

Pure 2*m*-pole Field: Cos(*m*θ) Distribution

For any point, *P*, inside the current shell:

$$\boldsymbol{B}_{in}(z) = -\left(\frac{\mu_0 I_0}{2\pi a}\right)\int_0^{2\pi}\sum_{n=1}^{\infty}\left(\frac{z}{a}\right)^{n-1}\exp(-in\phi)\cos(m\phi)d\phi$$



$$\int_0^{2\pi}\cos(n\phi)\cos(m\phi)d\phi = \pi\delta_{mn}; \int_0^{2\pi}\sin(n\phi)\cos(m\phi)d\phi = 0$$

$$\boxed{\boldsymbol{B}_{in}(z) = -\left(\frac{\mu_0 I_0}{2a}\right)\left(\frac{z}{a}\right)^{m-1}}$$

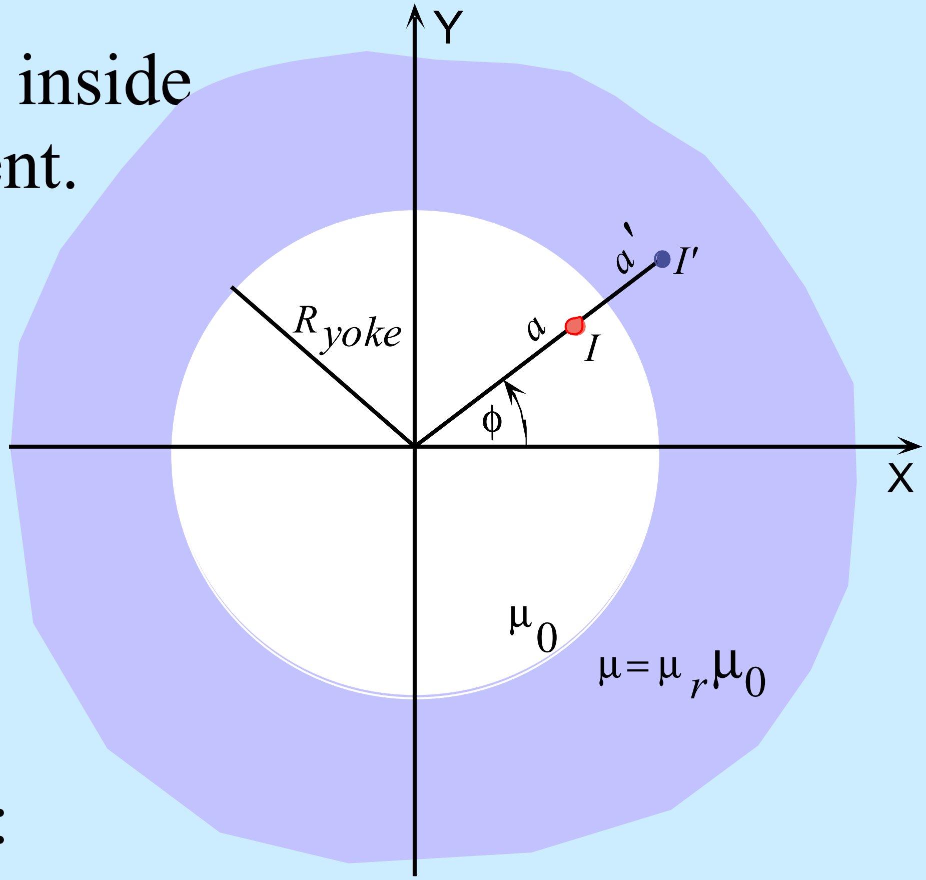
Pure **2*m***-pole Field

$$\boldsymbol{B}_{out}(z) = \left(\frac{\mu_0 I_0}{2\pi z}\right)\int_0^{2\pi}\left[1+\sum_{n=1}^{\infty}\exp(in\phi)\left(\frac{a}{z}\right)^{n}\right]\cos(m\phi)d\phi = \left(\frac{\mu_0 I_0}{2a}\right)\left(\frac{a}{z}\right)^{m+1}$$

Falls off outside as $1/r^{m+1}$

Current Filament Inside Cylindrical Cavity in Infinite Iron Yoke

The effect of iron yoke on the field inside can be described by an image current.



The location and strength of the image current is given by:

$$a' = \frac{R_{yoke}^2}{a}; \quad I' = \left(\frac{\mu_r - 1}{\mu_r + 1}\right) I; \quad \phi' = \phi$$

The $2n$-pole harmonic is given by:

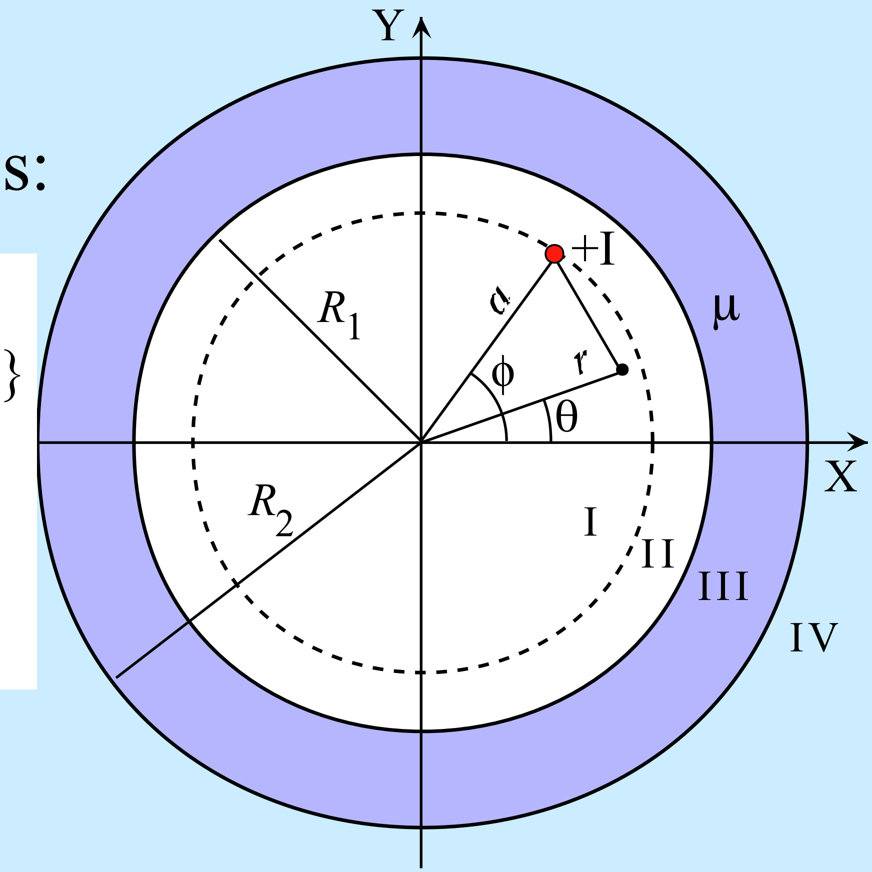
$$B_n + iA_n = -\left(\frac{\mu_0 I}{2\pi a}\right)\left(\frac{R_{ref}}{a}\right)^{n-1}\left[1 + \left(\frac{\mu_r - 1}{\mu_r + 1}\right)\left(\frac{a}{R_{yoke}}\right)^{2n}\right].\exp(-in\phi)$$

Current Filament Inside Cylindrical Shell

The general expansion of the vector potential in each of the four regions is:

$$A_z(r,\theta) = D_0 \ln\left(\frac{r}{a}\right) + \sum_{n=1}^{\infty} D_n \left(\frac{1}{r}\right)^n \cos\{n(\theta-\phi)\}$$

$$+\sum_{n=1}^{\infty} E_n r^n \cos\{n(\theta-\phi)\}$$



Boundary Conditions:

(a) The field components (derivatives of A_z) are finite everywhere.

(b) The radial component, B_r, is continuous at $r = a$; $r = R_1$ & $r = R_2$.

(c) The azimuthal component, H_θ, is continuous at $r = R_1$ & at $r = R_2$ [$H_\theta = B_\theta/(\mu\mu_0)$ in region III and $= B_\theta/\mu_0$ elsewhere.]

Current Filament Inside Cylindrical Shell

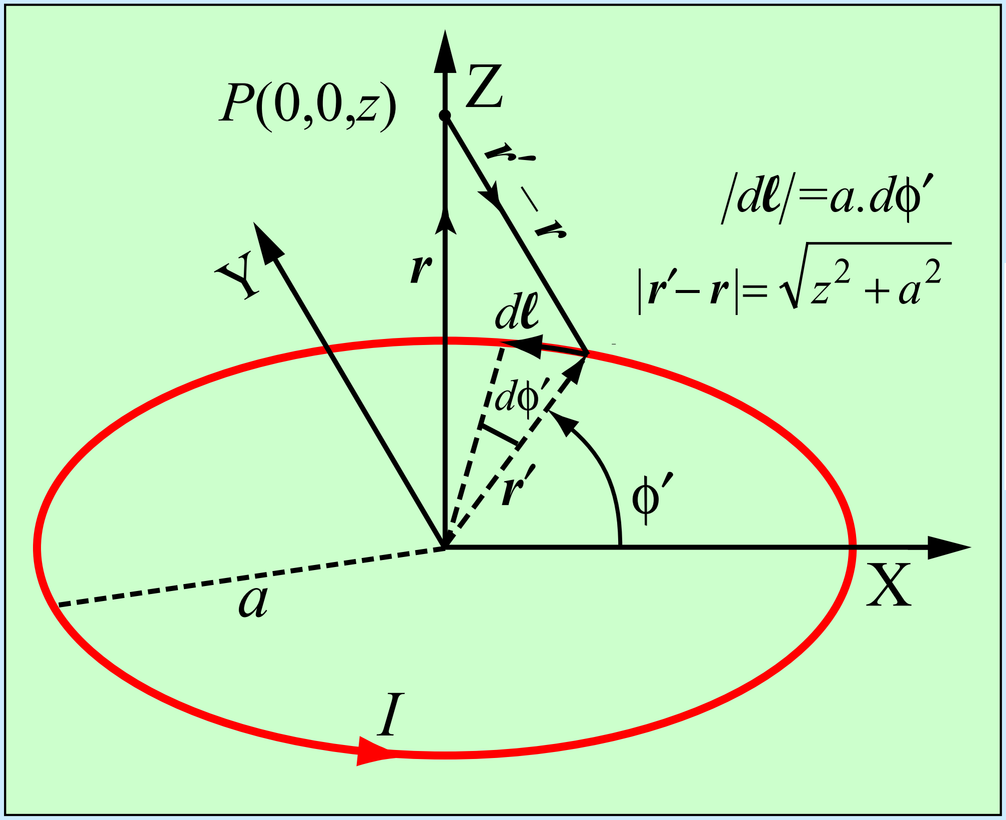
In Region I, which is of most interest:

$$A_z(r,\theta) = \left(\frac{\mu_o I}{2\pi}\right)\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)\left(\frac{r}{a}\right)^n \left[1 + \left(\frac{a}{R_1}\right)^{2n}\left(\frac{\mu-1}{\mu+1}\right)\frac{\left\{1-(R_1/R_2)^{2n}\right\}}{\left\{1-\left(\frac{\mu-1}{\mu+1}\right)^2 (R_1/R_2)^{2n}\right\}}\right]\cos\{n(\theta-\phi)\}$$

$$B_r(r,\theta) = -\left(\frac{\mu_o I}{2\pi a}\right)\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^{n-1} \left[1 + \left(\frac{a}{R_1}\right)^{2n}\left(\frac{\mu-1}{\mu+1}\right)\frac{\left\{1-(R_1/R_2)^{2n}\right\}}{\left\{1-\left(\frac{\mu-1}{\mu+1}\right)^2 (R_1/R_2)^{2n}\right\}}\right]\sin\{n(\theta-\phi)\}$$

$$B_\theta(r,\theta) = -\left(\frac{\mu_o I}{2\pi a}\right)\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^{n-1} \left[1 + \left(\frac{a}{R_1}\right)^{2n}\left(\frac{\mu-1}{\mu+1}\right)\frac{\left\{1-(R_1/R_2)^{2n}\right\}}{\left\{1-\left(\frac{\mu-1}{\mu+1}\right)^2 (R_1/R_2)^{2n}\right\}}\right]\cos\{n(\theta-\phi)\}$$

Field on the Axis of a Ring Current



$$\mathbf{A}(0,0,z)=\frac{\mu_0 I}{4\pi}\oint\frac{d\boldsymbol{\ell}}{|\boldsymbol{r}'-\boldsymbol{r}|}$$

$$\mathbf{B}(0,0,z)=\frac{\mu_0 I}{4\pi}\oint\frac{(\boldsymbol{r}'-\boldsymbol{r})\times d\boldsymbol{\ell}}{|\boldsymbol{r}'-\boldsymbol{r}|^3}$$

$$\boldsymbol{r}'-\boldsymbol{r}=(a\cos\phi')\hat{\boldsymbol{x}}+(a\sin\phi')\hat{\boldsymbol{y}}-z\hat{\boldsymbol{z}}$$

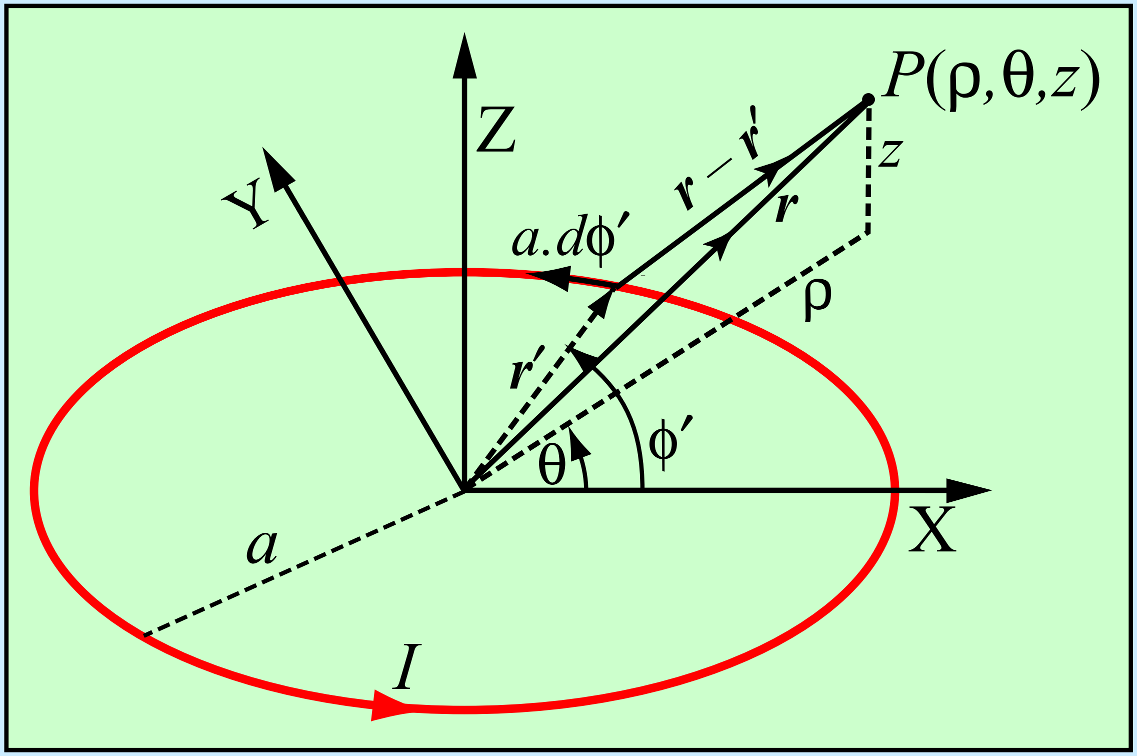
$$d\boldsymbol{\ell}=(-a\sin\phi')\hat{\boldsymbol{x}}+(a\cos\phi')\hat{\boldsymbol{y}}$$

$$|\boldsymbol{r}'-\boldsymbol{r}|=\sqrt{a^2+z^2}=\text{Constant}$$

$$B_x(0,0,z)=0;\quad B_y(0,0,z)=0$$

$$B_z(0,0,z)=\frac{\mu_0 I}{4\pi}\frac{a^2}{\left(a^2+z^2\right)^{3/2}}\int_0^{2\pi}d\phi'=\left(\frac{\mu_0 I}{2}\right)\frac{a^2}{\left(a^2+z^2\right)^{3/2}}$$

Field from a Ring Current: Off-Axis



$$\mathbf{A}(\boldsymbol{r}) \equiv A_\theta(\rho, z)\,\hat{\theta}$$

$$A_\theta = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{a\,d\phi'}{|\boldsymbol{r} - \boldsymbol{r}'|}$$

$$|\boldsymbol{r} - \boldsymbol{r}'| = \sqrt{\rho^2 + z^2 + a^2 - 2a\rho\cos(\phi' - \theta)}$$

$$k^2 = \frac{4a\rho}{(a+\rho)^2 + z^2}; \ (k^2 \le 1)$$

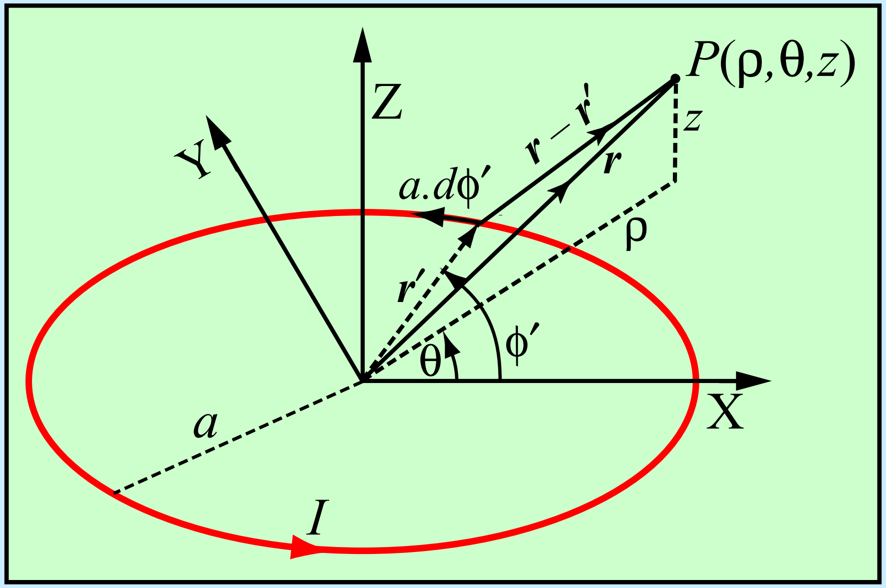
$$A_\theta(\rho, z) = \frac{\mu_0 I}{k\pi}\left(\frac{a}{\rho}\right)^{1/2}\left[\left(1 - \frac{k^2}{2}\right)K(k) - E(k)\right]$$

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}}$$

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2\theta}\; d\theta$$

Elliptic Integrals

Field from a Ring Current: Off-Axis



$$k^2 = \frac{4a\rho}{(a+\rho)^2 + z^2}; \; (k^2 \le 1)$$

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

$$E(k) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta}\, d\theta$$

Elliptic Integrals

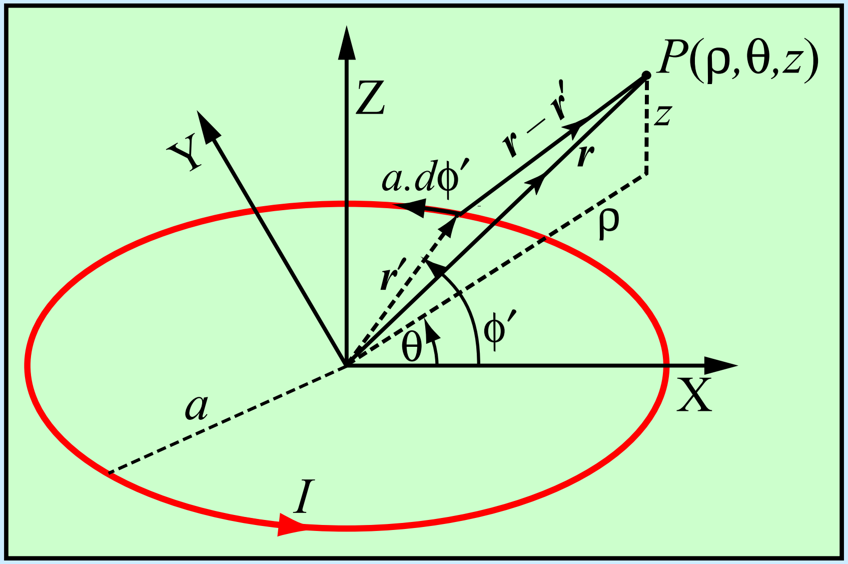
$$A_\theta(\rho,z) = \frac{\mu_0 I}{k\pi}\left(\frac{a}{\rho}\right)^{1/2}\left[\left(1-\frac{k^2}{2}\right)K(k) - E(k)\right]$$

$$B_\rho(\rho,z) = \frac{\mu_0 I}{2\pi}\left(\frac{z}{\rho}\right)\frac{1}{\left[(a+\rho)^2+z^2\right]^{1/2}}\left[-K(k) + \frac{a^2+\rho^2+z^2}{(a-\rho)^2+z^2}E(k)\right]$$

$$B_z(\rho,z) = \frac{\mu_0 I}{2\pi}\frac{1}{\left[(a+\rho)^2+z^2\right]^{1/2}}\left[K(k) + \frac{a^2-\rho^2-z^2}{(a-\rho)^2+z^2}E(k)\right]$$

Care needed to calculate A_θ and B_ρ for $\rho \to 0$

Field from a Ring Current: Near-Axis



$$k^2 = \frac{4a\rho}{(a+\rho)^2 + z^2} \to 0$$

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \approx \frac{\pi}{2}\left[1 + \frac{k^2}{4} + \frac{9k^4}{64} + \cdots\right]$$

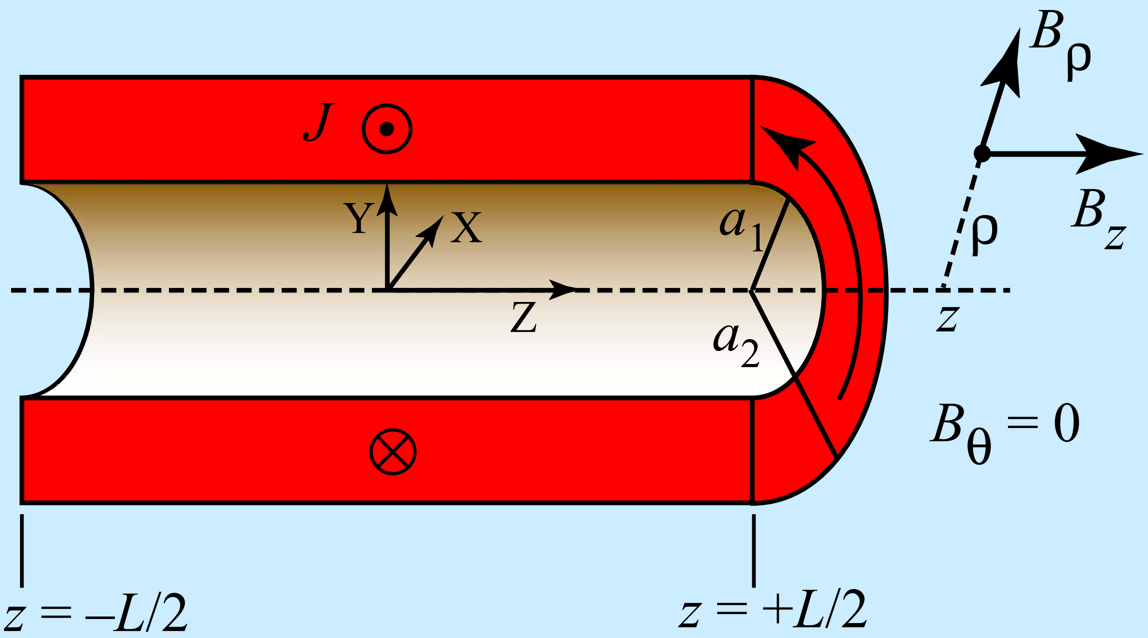
$$E(k) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta}\; d\theta \approx \frac{\pi}{2}\left[1 - \frac{k^2}{4} - \frac{3k^4}{64} + \cdots\right]$$

$$A_\theta(\rho,z) = \frac{\mu_0 I}{k\pi}\left(\frac{a}{\rho}\right)^{1/2}\left[\left(1-\frac{k^2}{2}\right)K(k) - E(k)\right] \xrightarrow[\rho\to 0]{} \left(\frac{\mu_0 I}{4}\right)\frac{a^2\rho}{\left[(a+\rho)^2 + z^2\right]^{3/2}}$$

$$B_\rho(\rho,z) = -\left(\frac{dA_\theta}{dz}\right) \xrightarrow[\rho\to 0]{} \left(\frac{3\mu_0 I}{4}\right)\frac{a^2\rho z}{\left[(a+\rho)^2 + z^2\right]^{5/2}}$$

$$B_z(\rho,z) = \frac{1}{\rho}\frac{d}{d\rho}(\rho A_\theta) \xrightarrow[\rho\to 0]{} \left(\frac{\mu_0 I}{2}\right)\frac{a(a-\rho)}{\left[(a+\rho)^2 + z^2\right]^{1/2}\;\left[(a-\rho)^2 + z^2\right]}$$

Field from a Solenoid



Field due to a solenoid can be obtained by integrating the field due to a thin ring.

The field components for off-axis points can be reduced to a 1-dimensional integral.

$$B_z(0,0,z) = \left(\frac{\mu_0 J}{2}\right) \int_{a_1}^{a_2} \int_{-L/2}^{L/2} \frac{a^2 \cdot da \cdot dz'}{\left(a^2 + (z - z')^2\right)^{3/2}}$$

On-axis field can be obtained in a closed form.

On-axis field is purely axial

$$B_z(0,0,z) = \left(\frac{\mu_0 J}{2}\right)\left[\left(\frac{L}{2} - z\right)\ln\left\{\frac{a_2 + \sqrt{a_2^2 + (L/2 - z)^2}}{a_1 + \sqrt{a_1^2 + (L/2 - z)^2}}\right\} + \left(\frac{L}{2} + z\right)\ln\left\{\frac{a_2 + \sqrt{a_2^2 + (L/2 + z)^2}}{a_1 + \sqrt{a_1^2 + (L/2 + z)^2}}\right\}\right]$$