Transformation Relations for Harmonic Coefficients

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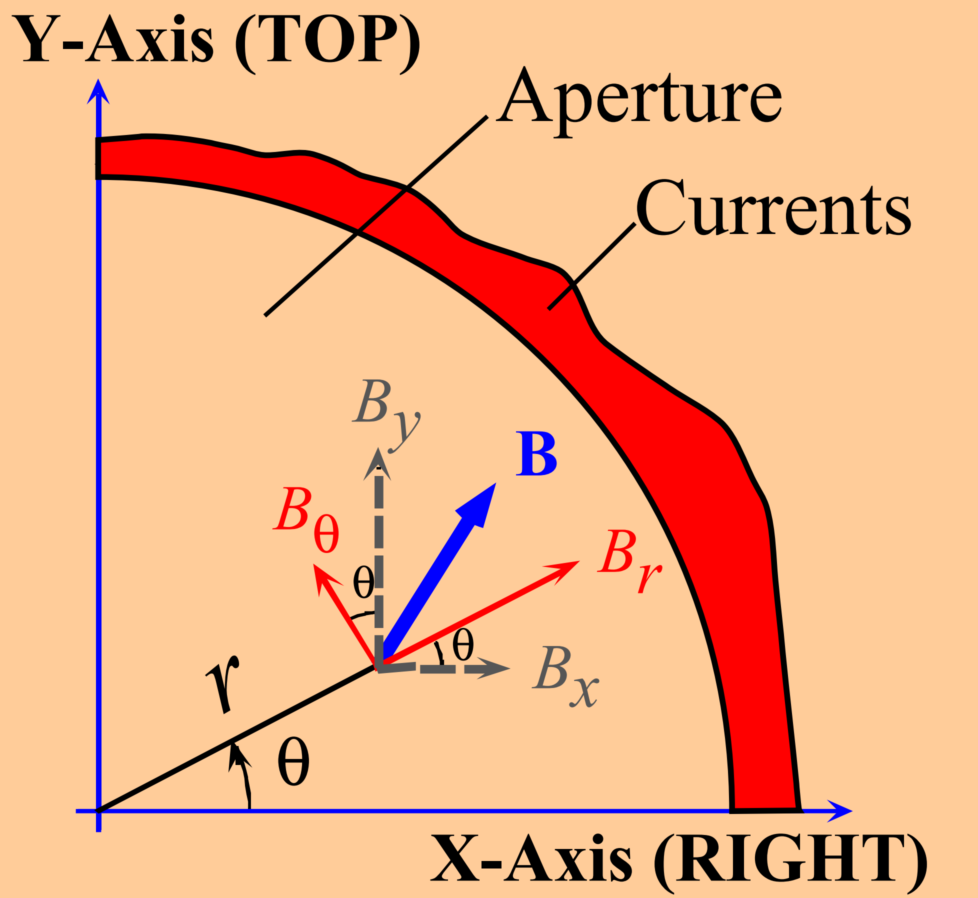
Commonly Used Coordinate System



View from the Lead End of the Magnet

Users of magnetic measurements data may use a system oriented differently, often requiring suitable transformations of the measured harmonics.

$$B_x(r,\theta) = B_r \cos\theta - B_\theta \sin\theta$$

$$B_y(r,\theta) = B_r \sin\theta + B_\theta \cos\theta$$

2-D Fields: Harmonic Series

- Components of 2-D fields in cylindrical coordinates:

$$B_r(r,\theta) = \sum_{n=1}^{\infty} C(n) \left(\frac{r}{R_{ref}} \right)^{n-1} \sin[n(\theta - \alpha_n)]$$

$$B_\theta(r,\theta) = \sum_{n=1}^{\infty} C(n) \left(\frac{r}{R_{ref}} \right)^{n-1} \cos[n(\theta - \alpha_n)]$$

- $C(n)$ = ***Amplitude***, α_n = ***phase angle*** of the ***2n-pole term*** in the expansion.
- R_{ref} = ***Reference radius***, arbitrary, typically chosen ~ the region of interest.

$\boxed{\text{C(n) scales as } R_{ref}^{n-1}}$

2-D Fields: Cartesian Components

- Cartesian components of **B** may be written as:

$$B_x(r,\theta) = \sum_{n=1}^{\infty} C(n) \left(\frac{r}{R_{ref}} \right)^{n-1} \sin[(n-1)\theta - n\alpha_n]$$

$$B_y(r,\theta) = \sum_{n=1}^{\infty} C(n) \left(\frac{r}{R_{ref}} \right)^{n-1} \cos[(n-1)\theta - n\alpha_n]$$

- A ***Complex field***, $\boldsymbol{B}(\boldsymbol{z}) = B_y + iB_x$, where $z = x + iy$, combines the 2 equations above:

$$\boxed{\boldsymbol{B}(\boldsymbol{z}) = \sum_{n=1}^{\infty} [C(n)\exp(-in\alpha_n)] \left(\frac{\boldsymbol{z}}{R_{ref}} \right)^{n-1}}$$

2-D Fields: Normal & Skew Terms

$$\boldsymbol{B}(z) = B_y + iB_x = \sum_{n=1}^{\infty} [C(n)\exp(-in\alpha_n)] \left(\frac{z}{R_{ref}} \right)^{n-1}$$

may be written as:

$$\boldsymbol{B}(z) = \sum_{n=1}^{\infty} [B_n + iA_n] \left(\frac{z}{R_{ref}} \right)^{n-1}$$

Simple power series, valid within source-free zone.

where:

$$B_n \equiv C(n)\cos(n\alpha_n) = 2n\text{ - pole NORMAL Term}$$

$$A_n \equiv -C(n)\sin(n\alpha_n) = 2n\text{ - pole SKEW Term}$$

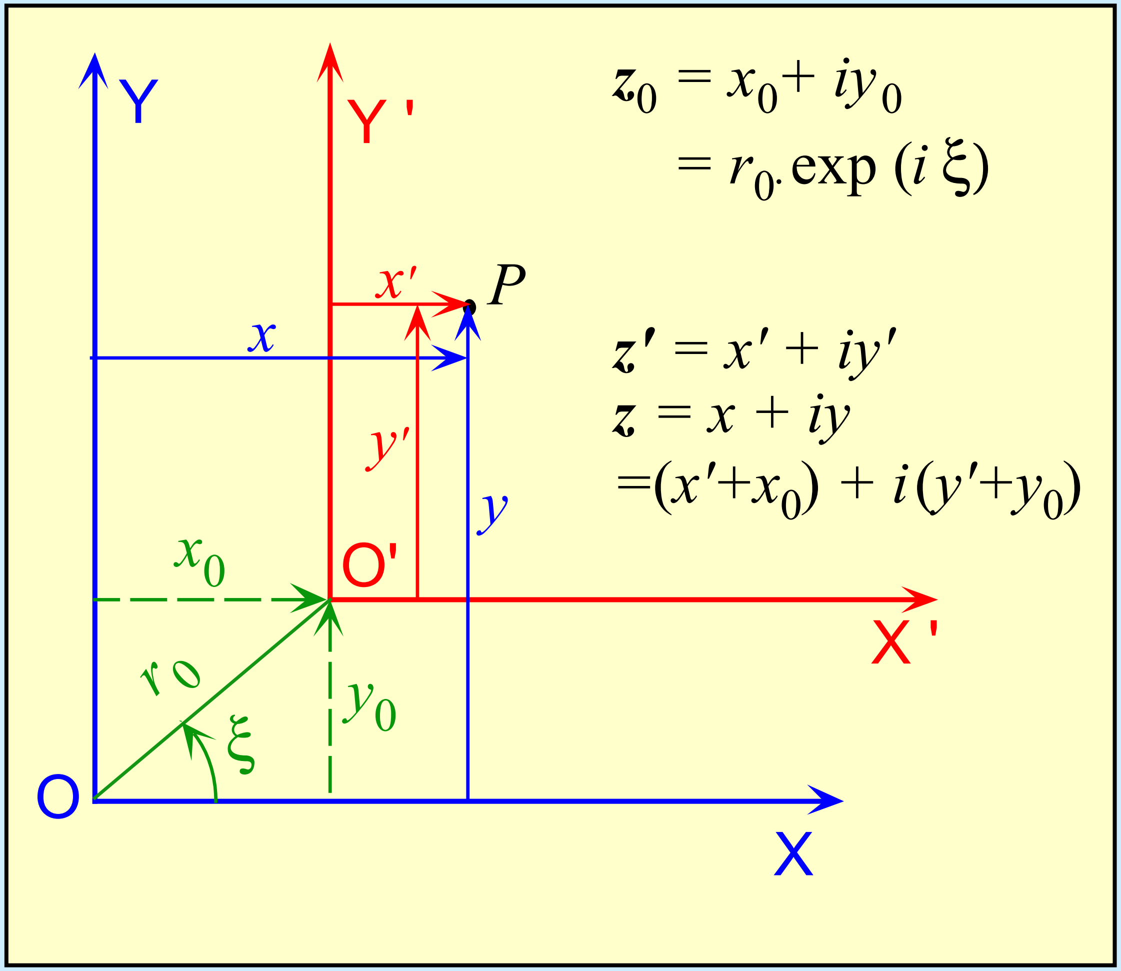
In the US, the 2*n*-pole terms are denoted by B_{n-1} and A_{n-1}.

Sometimes, the skew terms are defined without the negative sign, but the above form is the most common now.

Properties of Harmonics

- The Normal and Skew harmonics represent ***coefficients of expansion*** in a power series for the field components.
- The harmonics allow computation of field everywhere in the aperture (within a circle of convergence) using only a few numbers.
- These coefficients obviously depend on the choice of origin and orientation of the coordinate system. Measured harmonics, therefore, often need to be ***centered*** and ***rotated***.

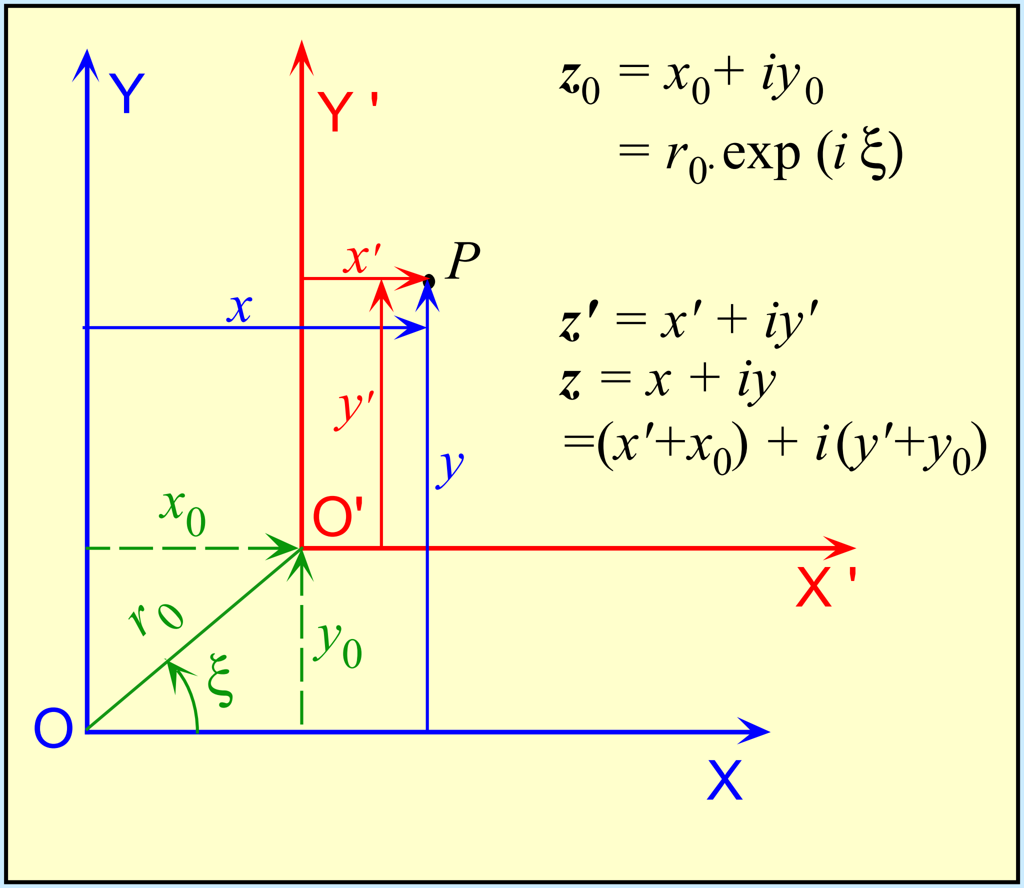
Centering of Harmonics: Definitions



X′-Y′ is a coordinate system displaced with respect to the X-Y frame by x_0 along X-axis and by y_0 along the Y-axis.

Field at the point P expressed as a function of (x,y) gives harmonics in the X-Y frame. Same field expressed in (x′,y′) gives harmonics in the X′-Y′ frame.

Centering of Harmonics



$$B(z') = B_{y'} + iB_{x'} = B_y + iB_x$$

$$= \sum_{n=n_0}^{\infty} (B_n + iA_n) \left(\frac{z}{R_{ref}} \right)^{n-n_0}$$

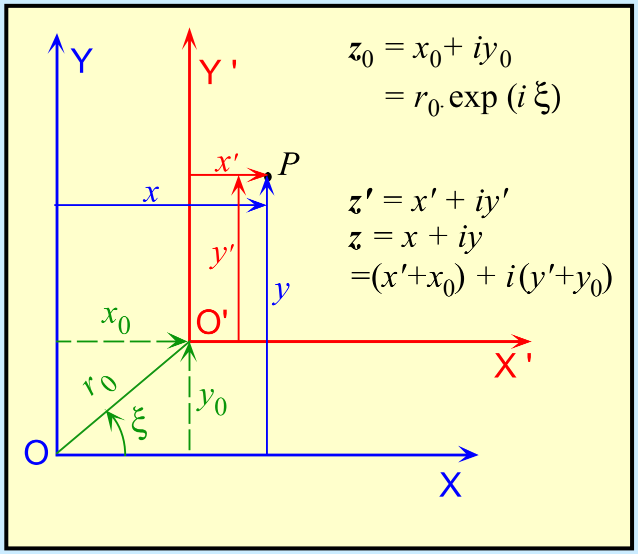
$$\equiv \sum_{n=n_0}^{\infty} (B'_n + iA'_n) \left(\frac{z'}{R_{ref}} \right)^{n-n_0}$$

$n_0 = 0$: US
$n_0 = 1$: European

(B_n, A_n) = Harmonics in X-Y frame

(B'_n, A'_n) = Harmonics in X'-Y' frame

Centering of Harmonics



$$\boldsymbol{B}(z') = B_{y'} + iB_{x'} = \sum_{n=n_0}^{\infty}(B_n' + iA_n')\left(\frac{z'}{R_{ref}}\right)^{n-n_0} = B_y + iB_x$$

$$= \sum_{k=n_0}^{\infty}(B_k + iA_k)\left(\frac{z}{R_{ref}}\right)^{k-n_0} = \sum_{k=n_0}^{\infty}(B_k + iA_k)\left(\frac{z' + z_0}{R_{ref}}\right)^{k-n_0}$$

$$= \sum_{k=n_0}^{\infty}(B_k + iA_k)\sum_{n=n_0}^{k}\frac{(k-n_0)!}{(n-n_0)!(k-n)!}\left(\frac{z'}{R_{ref}}\right)^{n-n_0}\left(\frac{z_0}{R_{ref}}\right)^{k-n}$$

$$\sum_{k=n_0}^{\infty}\sum_{n=n_0}^{k} t_{kn} = \sum_{n=n_0}^{\infty}\sum_{k=n}^{\infty} t_{kn}$$

$$= \sum_{n=n_0}^{\infty}\left[\sum_{k=n}^{\infty}(B_k + iA_k)\frac{(k-n_0)!}{(n-n_0)!(k-n)!}\left(\frac{z_0}{R_{ref}}\right)^{k-n}\right]\left(\frac{z'}{R_{ref}}\right)^{n-n_0}$$

$$(B_n' + iA_n') = \sum_{k=n}^{\infty}(B_k + iA_k)\left[\frac{(k-n_0)!}{(n-n_0)!(k-n)!}\right]\left(\frac{x_0 + iy_0}{R_{ref}}\right)^{k-n} \quad ;\, n \geq n_0$$

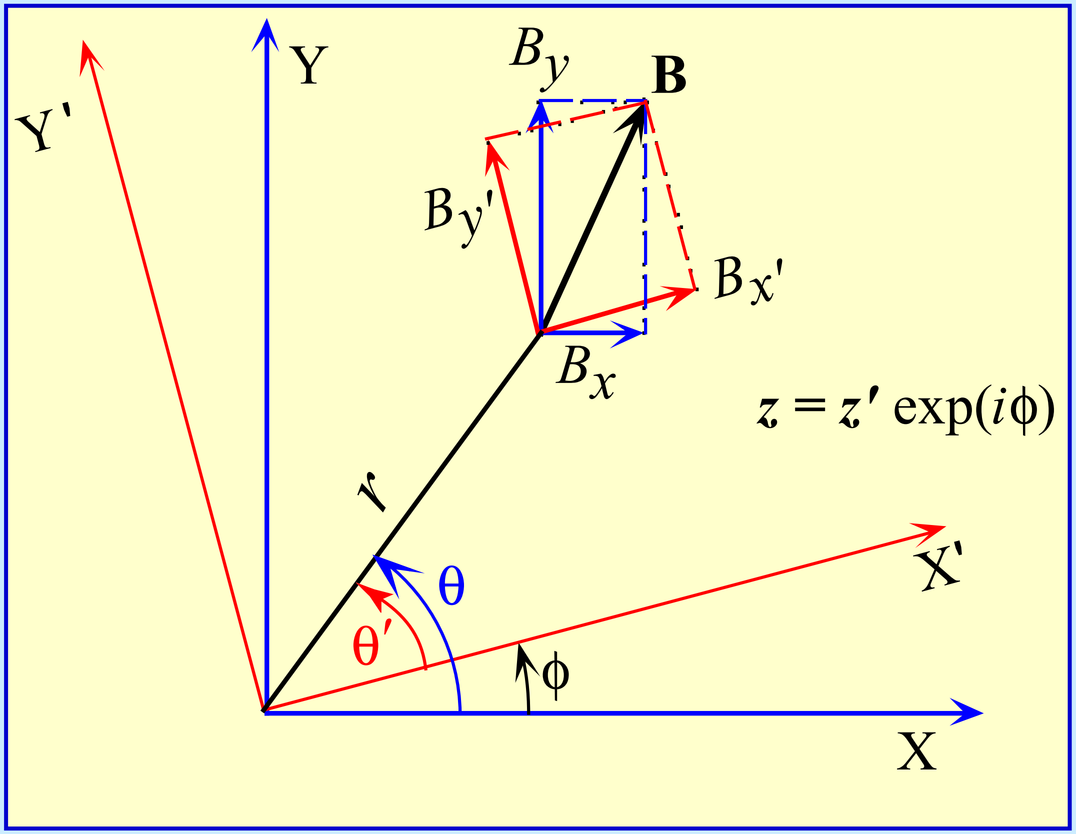
$n_0 = 0$: US
$n_0 = 1$: European

All higher harmonics contribute to a given harmonic in the displaced frame. This is referred to as FEED DOWN of harmonics

Centering of Harmonics

- If X-Y is the measurement frame, and X′-Y′ is the magnet frame, then one can compute the “centered” harmonics, provided the offsets (x_0, y_0) are known.
- For magnets other than dipoles, the magnetic center is defined as the point where the harmonic terms immediately below the main harmonic are zero. For example, the dipole terms should be zero in a quadrupole.
- For dipoles, higher order unallowed terms are made zero. Another technique involves powering the dipole in a quadrupole mode to determine (x_0, y_0)

Rotation of Harmonics: Definitions

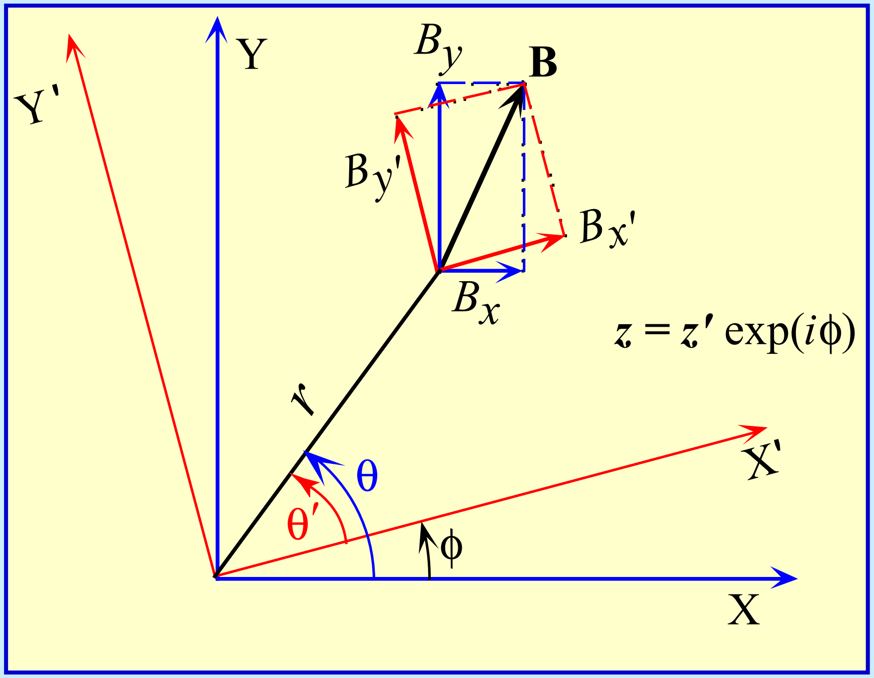


$$z = x + iy = r.\exp(i\theta)$$
$$z' = x' + iy' = r.\exp(i\theta')$$

X′-Y′ is a coordinate system rotated counterclockwise with respect to the X-Y frame by an angle ϕ.

Field at any point, expressed as a function of (x,y) gives harmonics in the X-Y frame. Same field expressed in (x′,y′) gives harmonics in the rotated X′-Y′ frame.

Rotation of Harmonics



$$B_{x'} = B_x \cos\phi + B_y \sin\phi$$

$$B_{y'} = -B_x \sin\phi + B_y \cos\phi$$

$$\boldsymbol{B}(z') = B_{y'} + iB_{x'}$$

$$= (B_y + iB_x)\exp(i\phi)$$

$$\boxed{(B'_n + iA'_n) = (B_n + iA_n)\exp[i(n - n_0 + 1)\phi];\ n \geq n_0}$$

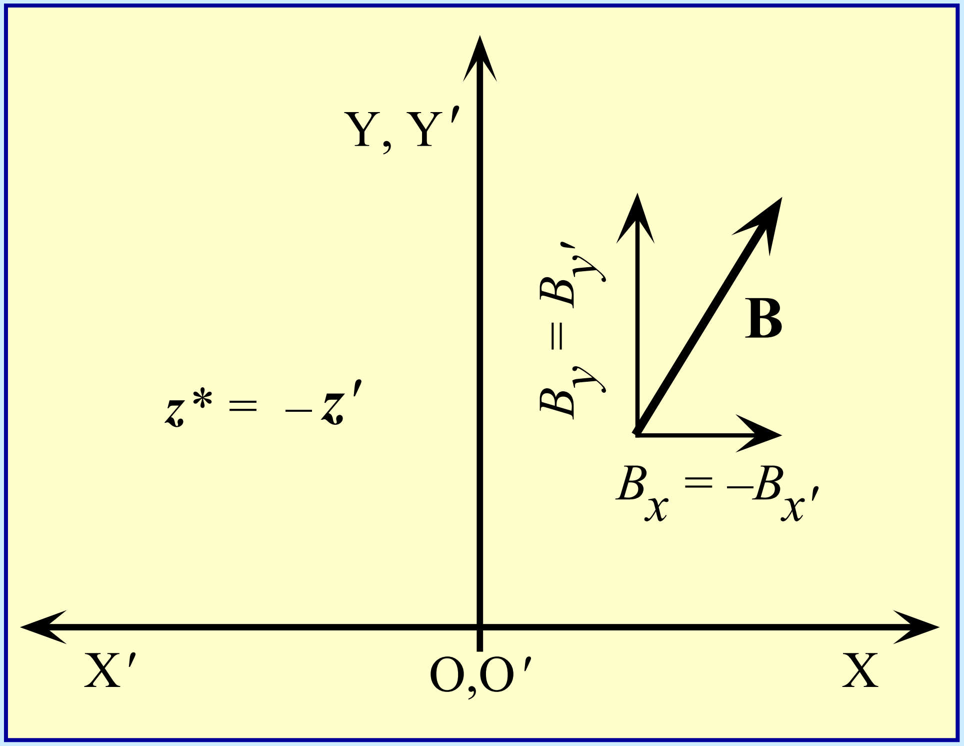
$n_0 = 0$: US; $n_0 = 1$: European

Rotation of frame causes mixing of normal and skew terms of the same harmonic. There is NO FEED DOWN with rotation.

Rotation of Harmonics

- The harmonics are often expressed in a reference frame aligned to the principle axes of a magnet.
- If X-Y is the measurement frame, and X′-Y′ is the magnet frame, then one can compute the "rotated" harmonics, provided the angle ϕ is known.
- The angle ϕ is chosen such that for the main harmonic, the skew term (in a Normal magnet) or the normal term (in a Skew magnet) is zero.

Reflection of X-axis



$$z^* = x - iy = -x' - iy' = -z'$$

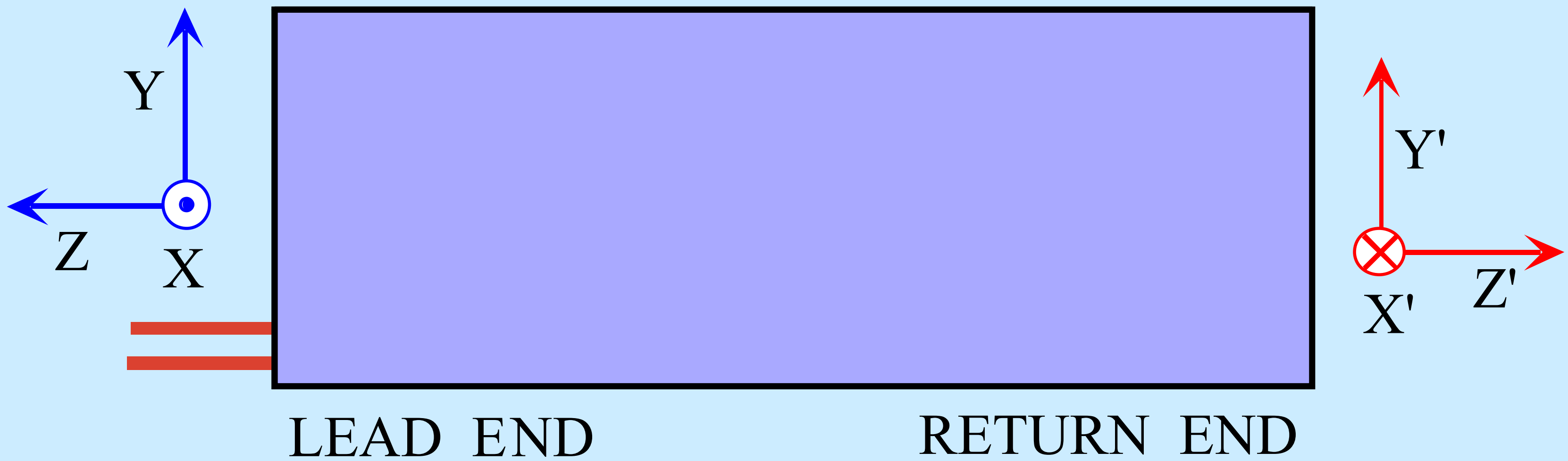
$$\begin{aligned}
B_{y'} + iB_{x'} &= B_y - iB_x = (B_y + iB_x)^* \\
&= \sum_{n=n_0}^{\infty} (B_n + iA_n)^* \left(\frac{z^*}{R_{ref}}\right)^{n-n_0} \\
&= \sum_{n=n_0}^{\infty} (B_n - iA_n)(-1)^{n-n_0} \left(\frac{z'}{R_{ref}}\right)^{n-n_0} \\
&\equiv \sum_{n=n_0}^{\infty} (B'_n + iA'_n) \left(\frac{z'}{R_{ref}}\right)^{n-n_0}
\end{aligned}$$

$$B'_n = (-1)^{n-n_0} B_n \,; \quad A'_n = (-1)^{n-n_0+1} A_n \,; \quad n \geq n_0$$

$n_0 = 0$: US; $n_0 = 1$: European

This transformation is useful in deriving conversion of harmonics measured from one end of the magnet to those that would be measured from the opposite end.

Harmonics from the Opposite End



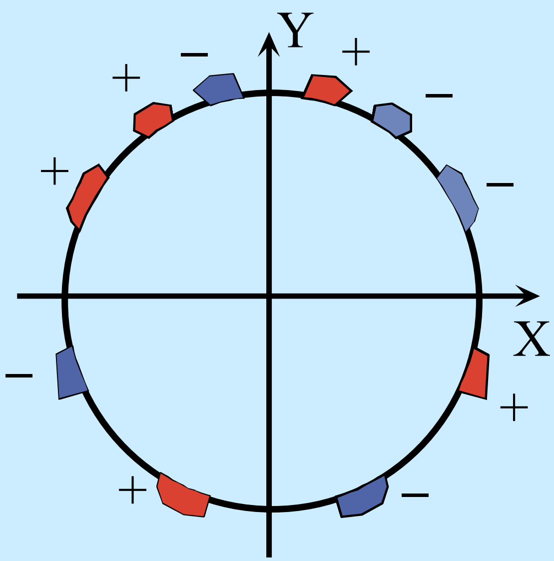
When viewed from the other end, the new X-axis (X') points in the opposite direction. The transformation for X-axis reflection is therefore applicable for this change.

Often, the harmonics are quoted with the magnet excitation polarity such that the most dominant term is positive. This may require (e.g. for normal quadrupole) that the polarity be changed when viewed from the opposite end. This amounts to <u>an extra change in sign</u> of all the harmonics in such cases.

Symmetries & Allowed Harmonics

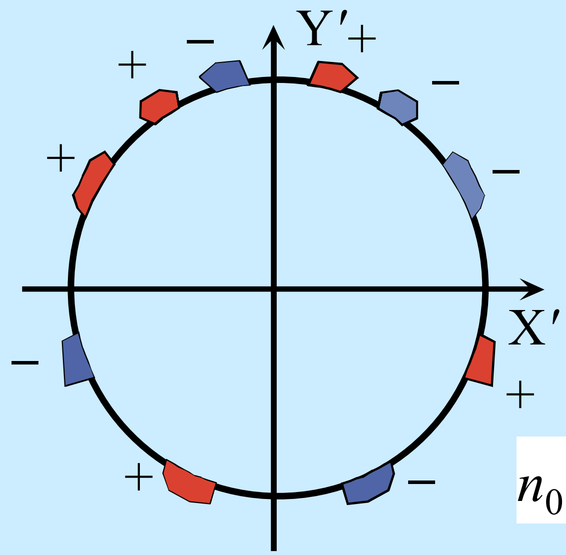
- Accelerator magnets often have definite symmetries (or antisymmetries) in the current distribution. These symmetries lead to only certain harmonics being ***allowed***.
- The coordinate transformations derived for harmonics may be used to identify the allowed terms. This approach does not require explicit knowledge of the relation between currents and harmonics.

Left-Right Antisymmetry in Current



LEAD END VIEW

(+ is current in the positive Z direction, which is from the Return End to the Lead End.)



RETURN END VIEW

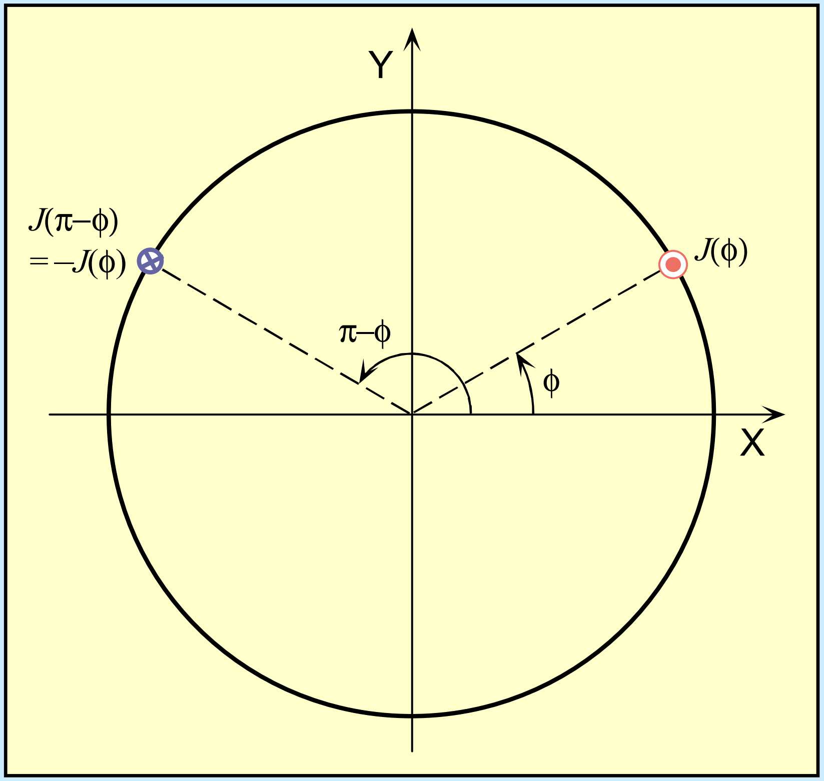
(+ is current in the NEW positive Z direction, which is from the Lead End to the Return End.)

$$B'_n = (-1)^{n-1} B_n$$
$$A'_n = (-1)^n A_n \; ;$$
$$(n \geq 1)$$

$n_0 = 1$ (European) Used

All harmonics <u>must remain the same</u> when viewed from the Return End. This implies only ODD NORMAL (dipole, sextupole, etc.) and EVEN SKEW terms are allowed.

L-R Antisymmetry: Another way



$$J(\pi-\phi)=-J(\phi)$$

$$B_n+iA_n \propto \int_0^{2\pi} J(\phi)e^{in\phi}d\phi = \int_{-\pi/2}^{\pi/2} J(\phi)\left[e^{in\phi}-e^{in(\pi-\phi)}\right]d\phi$$

$$= \int_{-\pi/2}^{\pi/2} J(\phi)\left[e^{in\phi}-(-1)^n e^{-in\phi}\right]d\phi$$

$$B_n+iA_n \propto i\int_{-\pi/2}^{\pi/2} J(\phi)\sin(n\phi)\,d\phi \quad \text{for EVEN } n$$

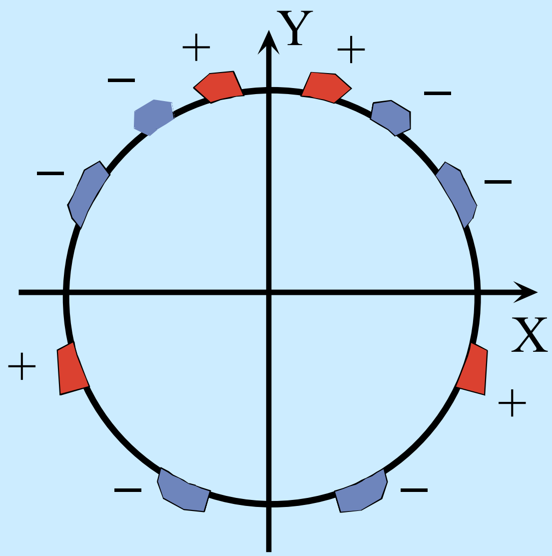
Real part = B_n = 0 for even n
Only ODD normal terms are allowed

$$B_n+iA_n \propto \int_{-\pi/2}^{\pi/2} J(\phi)\cos(n\phi)\,d\phi \quad \text{for ODD } n$$

Imaginary part = A_n = 0 for odd n
Only EVEN skew terms are allowed

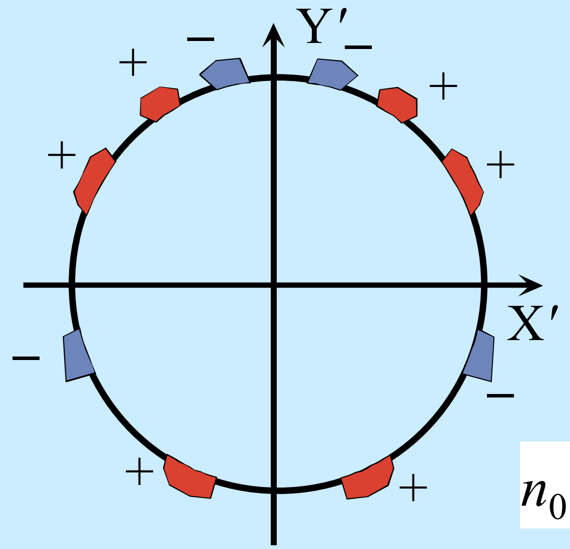
For every current element on the right half ($-\pi/2 \le \phi \le +\pi/2$), there is a corresponding current element on the left half of opposite sign.

Left-Right Symmetry in Current



LEAD END VIEW

(+ is current in the positive Z direction, which is from the Return End to the Lead End.)



RETURN END VIEW

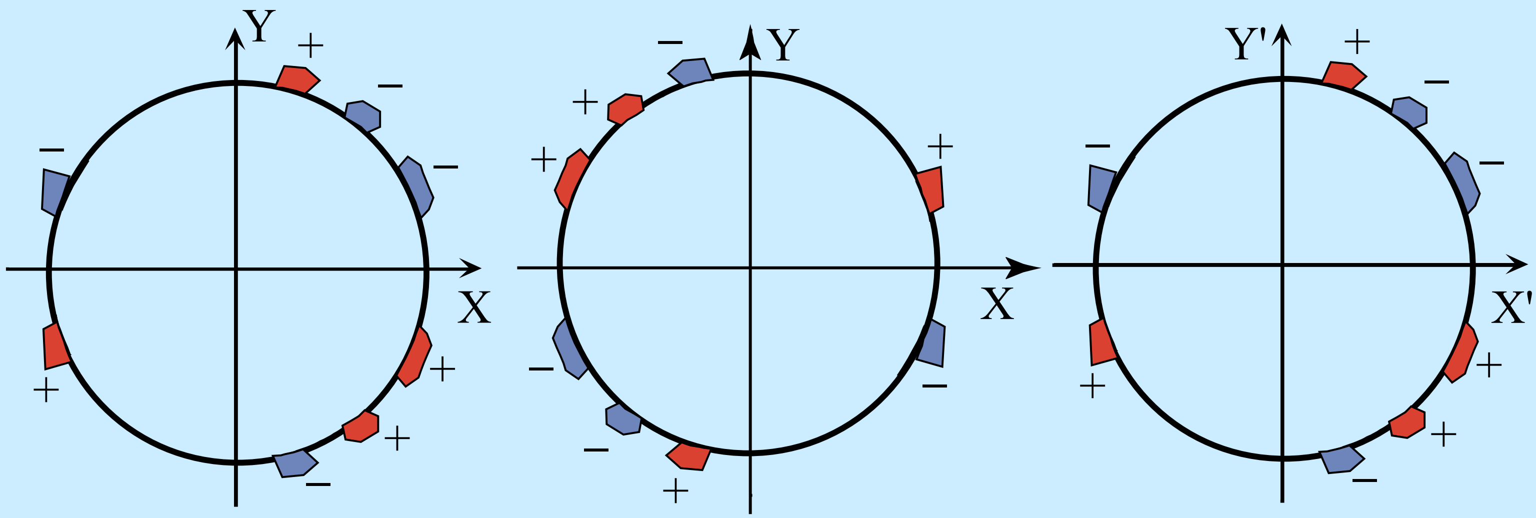
(+ is current in the NEW positive Z direction, which is from the Lead End to the Return End.)

$$B_n' = (-1)^{n-1} B_n$$
$$A_n' = (-1)^n A_n \; ;$$
$$(n \geq 1)$$

$n_0 = 1$ (European) Used

All harmonics <u>must change sign</u> when viewed from the Return End. This implies only EVEN NORMAL (Quadrupole, Octupole, etc.) and ODD SKEW terms are allowed.

Top-Bottom Antisymmetry in Current



$$B'_n = -B_n$$
$$A'_n = A_n \ ;$$
$$(n \geq 1)$$

$n_0 = 1$

(European)

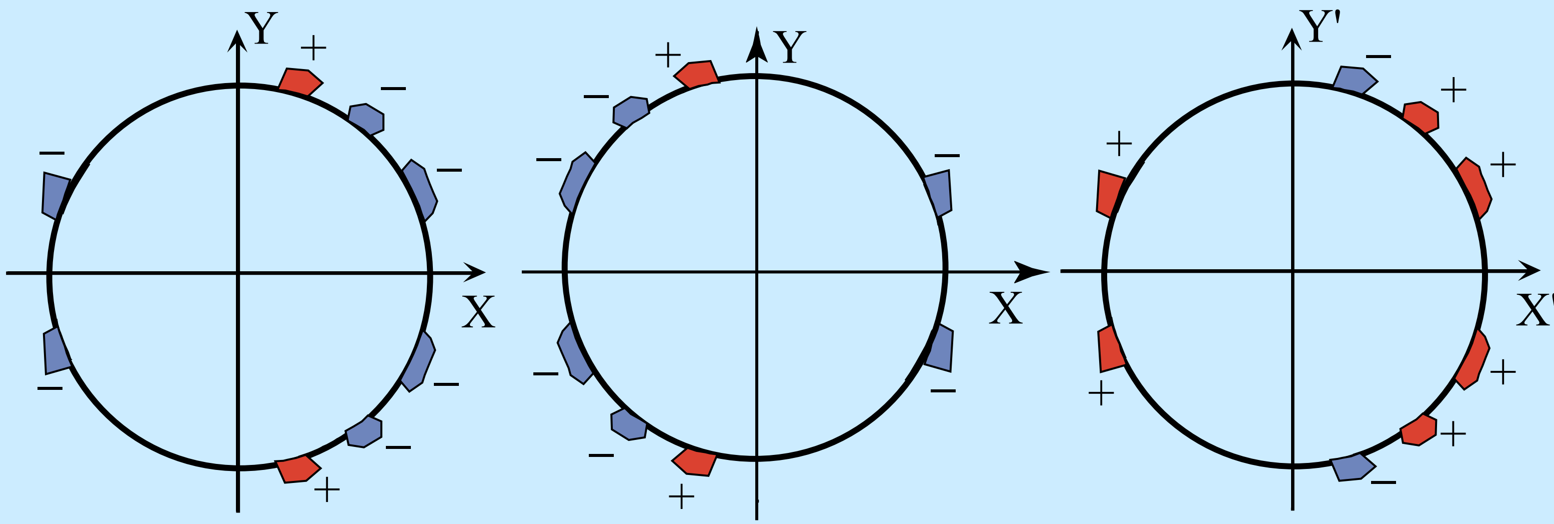
LEAD END VIEW → LE; 180° ROTATION → RETURN END VIEW

(+ is current in the positive Z direction, which is from the Return End to the Lead End.)

(+ is current in the NEW positive Z direction, which is from the Lead End to the Return End.)

All harmonics <u>must remain the same</u> when the magnet is rotated by 180 degrees and then viewed from the Return End. This implies only SKEW terms are allowed. All normal terms must be zero.

Top-Bottom Symmetry in Current



LEAD END VIEW ⟶ LE; 180° ROTATION ⟶ RETURN END VIEW

$$B'_n = -B_n$$
$$A'_n = A_n \;;$$
$$(n \geq 1)$$

$n_0 = 1$

(European)

(+ is current in the positive Z direction, which is from the Return End to the Lead End.)

(+ is current in the NEW positive Z direction, which is from the Lead End to the Return End.)

All harmonics <u>must change sign</u> when the magnet is rotated by 180 degrees and then viewed from the Return End. This implies only NORMAL terms are allowed. All skew terms must be zero.

Allowed Harmonics in a 2*m*-pole Magnet

A 2*m*-pole magnet has poles of opposite polarity at every (π/m) radians. All harmonics, therefore, must change sign when the magnet is rotated by (π/m) radians.

$$B_n' + iA_n' = (B_n + iA_n)\left[\cos\left(\frac{n\pi}{m}\right) + i\sin\left(\frac{n\pi}{m}\right)\right] = -(B_n + iA_n)$$

For non-zero harmonics, this implies that:

$n_0 = 1$ (European)

$$\boxed{\cos\left(\frac{n\pi}{m}\right) = -1; \text{ OR } \quad n = (2k+1)m;\ k = 0, 1, 2, 3, \ldots}$$

The only allowed harmonics in a 2*m*-pole magnet (with perfect 2*m*-fold antisymmetry) are those which are ODD MULTIPLES OF *m*. Unallowed harmonics appear in a magnet from a loss of symmetry due to construction errors.

Summary

- Within reasonable limits, the harmonics measuring system need not be centered or aligned to the axis of the magnet. The data can be ***centered*** and ***rotated*** using the appropriate transformations.
- The harmonic transformation laws are useful in applying the results of magnetic measurements to as-installed magnets in a machine, and for analyzing the impact of alignment errors.

Summary (Contd.)

- The harmonic transformation laws under various coordinate transformations can be used to derive ***allowed harmonics*** under various types of current symmetry, without using any explicit knowledge of relationship between currents and harmonics.
- Unallowed harmonics in a magnet are a result of violating the symmetry due to construction ***tolerances***. Unusually large unallowed harmonics point to construction ***mistakes***.