1.5. Magnetic Field Analysis in Accelerator Magnets

In this section an outline of the formalism and theory used in carrying out the field calculations in the superconducting magnets is given. Starting from first principles, basic expressions are developed which are used in designing and describing the magnetic fields in the accelerator magnets.

The uniformity of the magnetic field is very important since it determines the performance of the machine. A typical requirement for the field quality in the accelerator magnets is that the deviation from the ideal shape should be within a few parts in 10^4. The uniformity of the field is expressed in terms of the Fourier harmonic components.

1.5.1. Basic Electromagnetic Field Equations

The calculation of the magnetic field in accelerator magnets is too complex to be done directly by solving Maxwell's equations. However, the most complicated formulae describing the field shape in the magnets are derived primarily from them. In this section, Maxwell's equations and other commonly used expressions of electro-magnetic theory [95,129,150] are briefly described. Although the magnetic field in the accelerator magnets is not static in time, the effects of time variation are by and large negligible in the problems to be addressed during the course of this work. Therefore, most of the detailed analysis is limited to the magneto-static case only.

The four Maxwell's equations are :

$$\nabla \cdot \vec{D} = \rho, \tag{1.5.1a}$$

$$\nabla \cdot \vec{B} = 0, \tag{1.5.1b}$$

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0, \tag{1.5.1c}$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}. \tag{1.5.1d}$$

Here $\vec{H}$ is the magnetic field, $\vec{E}$ is the electric field, $\vec{B}$ is the magnetic induction and $\vec{D}$ is the displacement vector. ρ denotes the charge density and $\vec{J}$ the current density, and these two are related by the following continuity equation,

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0. \tag{1.5.2}$$

Furthermore, $\vec{B}$ and $\vec{H}$ are related by the following equations:

$$\frac{\vec{B}}{\mu_o} = \vec{H} + \vec{M}, \tag{1.5.3a}$$

$$\frac{\vec{B}}{\mu\mu_o} = \vec{H}, \tag{1.5.3b}$$

where μ_o is the permeability of the vacuum ($\mu_o = 4\pi \times 10^{-7}$ henry/meter) and μ is the relative permeability of the medium (relative with respect to that of vacuum). Often, μ is simply referred to as the permeability (which is in fact the case in CGS units) and the same convention is followed here unless otherwise explicitly mentioned. $\vec{M}$ denotes the magnetization (or magnetic polarization) of the medium. In free space (vacuum) $\vec{M}$ is 0. In an isotropic medium $\vec{H}$, $\vec{B}$ and $\vec{M}$ are parallel to each other.

Furthermore, $\vec{D}$ and $\vec{E}$ are related by the following equations:

$$\vec{D} = \epsilon_o \vec{E} + \vec{P}, \tag{1.5.4a}$$

$$\vec{D} = \epsilon\epsilon_o \vec{E}, \tag{1.5.4b}$$

where $\vec{P}$ is the electric polarization and ϵ_o is the permittivity in vacuum ($\epsilon_o = 8.854 \times 10^{-12}$ farad/meter). ϵ is the relative permittivity of the medium. In free space (vacuum), the electric polarization is 0.

The constants ϵ_o and μ_o are related through the relation

$$\epsilon_o \mu_o = \frac{1}{c^2},$$

where c is the velocity of light (c=2.998 $\times$ 10^8 m/s). Since $\vec{B}$ has a zero divergence, it may be expressed in term of a magnetic vector potential $\vec{A}$ as

$$\vec{B} = \nabla \times \vec{A}. \tag{1.5.5}$$

The vector potential $\vec{A}$ can be obtained at any point ($\vec{r}$) due to a current density $\vec{J}(\vec{r'})$ with the help of the following integral equation :

$$\vec{A}(\vec{r}) = \frac{\mu\mu_o}{4\pi} \int_V \frac{\vec{J}\left(\vec{r'}\right)}{|\vec{r} - \vec{r'}|} dv, \tag{1.5.6}$$

where $\vec{r}$ and $\vec{r'}$ are three dimensional coordinates and dv is the three dimensional volume element.

The components of the field in Eqs. (1.5.5) in Cartesian coordinates are given by

$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \tag{1.5.7a}$$

$$B_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \tag{1.5.7b}$$

$$B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_z}{\partial y}, \tag{1.5.7c}$$

and in cylindrical coordinates by

$$B_r = \frac{1}{r}\left(\frac{\partial A_z}{\partial \theta}\right) - \frac{\partial A_\theta}{\partial z}, \tag{1.5.7d}$$

$$B_\theta = \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}, \tag{1.5.7e}$$

$$B_z = \frac{1}{r}\left(\frac{\partial (rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right). \tag{1.5.7f}$$

The research work to be described is restricted to static magnetic fields only and electric fields are not considered. During the accelerating cycle of the machine, the magnetic field does change with time in the superconducting magnets. However, for the problems to be discussed during the course of this work, the change in magnetic field has negligible effect on field quality. Therefore the following two Maxwell's equations for the magnetostatic case are used in developing various formulae

$$\nabla \cdot \vec{B} = 0, \tag{1.5.8a}$$

$$\nabla \times \vec{H} = \vec{J}. \tag{1.5.8b}$$

Ampère's law

$$\oint_S \vec{H} \cdot ds = I, \tag{1.5.9}$$

can be obtained from Eqs. (1.5.8b) by integrating and using Stoke's theorem :

$$\oint_C \vec{V} \cdot d\vec{l} = \int_S \left(\nabla \times \vec{V}\right) \cdot \vec{n} da$$

where $\vec{V}$ is a well behaved vector field, S is an open arbitrary surface, C is the closed curve bounding S, d$\vec{l}$ is a line element of C, and $\vec{n}$ is a vector element normal to S. The right hand side of the equation simply states that I = $\int \vec{J} \cdot \vec{n} da$ is the total current flowing through the area.

Poisson's equation for the vector potential is derived here under the assumptions that $\vec{B} = \mu_0 \mu \vec{H}$, the medium is homogeneous (i.e. μ is constant over a finite space) and isotropic. Using $\vec{B} = \mu_0 \mu \vec{H}$ and $\vec{B} = \nabla \times \vec{A}$ in Eqs. (1.5.8), one obtains :

$$\nabla \times \nabla \times \vec{A} = \mu_0 \mu \vec{J}. \tag{1.5.10}$$

The following identity is used to simplify the above equation :

$$\nabla^2 \vec{A} = \nabla \left(\nabla \cdot \vec{A} \right) - \nabla \times \nabla \times \vec{A}. \tag{1.5.11}$$

In Cartesian coordinates the above Laplacian operator (∇^2) can be applied to a vector $\vec{A}$ whose i^{th} component is $\nabla^2 A_i$. In other coordinate systems Eq. (1.5.11) must be used to determine the expression for $\nabla^2 \vec{A}$. In the cylindrical coordinate system :

$$\nabla^2 \vec{A}_z = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial A_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 A_z}{\partial \theta^2}, \tag{1.5.12}$$

when $A_r = A_\theta = 0$ by symmetry (axial symmetric case).

The choice of $\nabla \cdot \vec{A}$ has thus far has been arbitrary and it is made zero in the Coulomb gauge (in the magnetostatic case). In that case Eq. (1.5.10) leads to Poisson's Equation as

$$\nabla^2 \vec{A} = -\mu_0 \mu \vec{J}. \tag{1.5.13}$$

In the 2-dimensional case, when the direction of current flow is parallel to the z-axis, $J_x = J_y = 0$. This implies that $A_x = A_y = 0$ and $\frac{\partial A_z}{\partial z} = 0$. Therefore, the above expression becomes,

$$\nabla^2 A_z = -\mu_0 \mu J_z, \tag{1.5.14}$$

which in the Cartesian coordinate system gives :

$$\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} = -\mu_0 \mu J_z. \tag{1.5.15}$$

In the case of axial symmetry, the Eq. (1.5.14) in cylindrical coordinates becomes :

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial A_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 A_z}{\partial \theta^2} = -\mu_0 \mu J_z, \tag{1.5.16}$$

on using Eq. (1.5.12).

1.5.2. Field Harmonic Definitions

It is useful to describe the magnetic field inside the aperture of accelerator magnets in terms of harmonic coefficients [96,140,144,175]. The discussion will be limited to 2-dimensional analysis, which describes the field in the body (or straight section) of a long magnet. When the magnetic field is evaluated a few aperture diameters away from the two ends of the magnet, the axial component of the field is negligible. The accelerator magnets examined here are those in which the field is consists of one fundamental harmonic which is several orders of magnitude larger (usually 10^4) than any other harmonic present.

The skew (a_n) and the normal (b_n) field harmonics are defined through the following relation :

$$B_y + iB_x = 10^{-4} B_{R_0} \sum_{n=0}^{\infty} [b_n + i\ a_n] [cos\,(n\theta) + i\ \ sin\,(n\theta)] \left(\frac{r}{R_0}\right)^n, \tag{1.5.17}$$

where B_x and B_y are the horizontal and vertical components of the magnetic field at (r,θ) and $i = \sqrt{-1}$. R_0 is the normalization radius. The magnets for the Relativistic Heavy Ion Collider (RHIC) have a coil radius ranging from 40 mm to 90 mm. In most of these magnets, the normalization radius is taken to be $\frac{5}{8}$ of the coil radius. The value of the normalization radius is 25 mm for the 80 mm aperture diameter of the RHIC arc dipoles and quadrupoles, 40 mm for the 130 mm aperture of the RHIC insertion quadrupoles, 31 mm for the 100 mm aperture of the RHIC insertion dipoles and 60 mm for the 180 mm aperture RHIC insertion dipoles [140]. B_{R_0} is the magnitude of the field due to the fundamental harmonic at the reference radius on the midplane. In the dipoles, $B_{R_0} = B_0$ (the field at the center of the magnet), in the quadrupoles, $B_{R_0} = G \times R_0$ (G being the field gradient at the center of the magnet), and in general for a $2(m+1)^{th}$ pole magnet,

$$B_{R_0} = \frac{R^m}{m!} \left[\frac{\partial^m B_y}{\partial x^m}\right]_{x=0,y=0}. \tag{1.5.18}$$

Eq. (1.5.17) can be re-written in several other forms using complex variables. In this section z represents the complex coordinate and $B(z)$ represents the complex field as follows:

$$z = x\ +\ i\ y,$$

$$(x\ +\ i\ y)^n = r^n\,(cos\,[n\theta]\ +\ i\ sin\,[n\theta]),$$

$$B\,(z) = B_y\ +\ i\ B_x,$$

$$c_n = b_n +\ i\ a_n,$$

Thus :

$$B_y + iB_x = 10^{-4} B_{R_0} \sum_{n=0}^{\infty} [b_n + i \; a_n] [x \; + \; i \; y]^n \left(\frac{1}{R_0}\right)^n \tag{1.5.19}$$

$$B(z) = 10^{-4} B_{R_0} \sum_{n=0}^{\infty} c_n \left(\frac{z}{R_0}\right)^n \tag{1.5.20}$$

The harmonics used so far (a_n, b_n, c_n) are all dimensionless coefficients. However, in another representation, the field is expressed in terms of coefficients which carry the units of magnetic field. These are usually distinguished from the harmonics a_n and b_n given in Eq. (1.5.17) by the use of the uppercase alphabet. The two are related as follows:

$$A_{n+1} = 10^{-4} B_{R_0} \; a_n, \tag{1.5.21a}$$

$$B_{n+1} = 10^{-4} B_{R_0} \; b_n, \tag{1.5.21b}$$

$$C_{n+1} = 10^{-4} B_{R_0} \; c_n. \tag{1.5.21c}$$

Using these, Eq. (1.5.20) can be written as :

$$B(z) = \sum_{n=1}^{\infty} C_n \left(\frac{z}{R_0}\right)^{n-1}. \tag{1.5.22}$$

In this case the summation begins from $n = 1$ instead of $n = 0$. Sometimes C_n is also written as $C(n)$.

The definition for the field harmonics used so far is the one which is more common in U.S. laboratories. The European laboratories (such as CERN and HERA) use a slightly different definition [179]. The two are related as follows :

$$(a_{n+1})_{Europe} = \; -10^{-4} \; (a_n)_{US}$$

$$(b_{n+1})_{Europe} = \; 10^{-4} \; (b_n)_{US}$$

Yet another representation of field harmonic is used in beam dynamics calculations where the particle trajectory is studied in the machine [25]. For this purpose, the field is expressed in the form of a Taylor series. The vertical component of the field on the median plane is expressed as

$$B_y(x,0) \;=\; \sum_{n=0}^{\infty} \frac{1}{n!} \left[\frac{d^n B_y}{dx^n}\right]_0 \; x^n, \tag{1.5.23}$$

where the subscript 0 implies that the derivatives are evaluated at the equilibrium orbit (which is generally at the center of the magnet). n=0 gives the vertical component of the field at the center of the magnet, which is represented as B_0 and the above equation becomes

$$B_y(x,0) \;=\; B_0 \;+\; \sum_{n=1}^{\infty} \frac{1}{n!} \left[\frac{d^n B_y}{dx^n}\right]_0 \; x^n, \tag{1.5.24}$$

Similarly, the horizontal component of the field (B_x) on the horizontal axis (X-axis) is expressed as :

$$B_x\,(x,0) = \sum_{n=0}^{\infty} \frac{1}{n!} \left[\frac{d^n B_x}{dx^n}\right]_0 \; x^n. \tag{1.5.25}$$

where, the subscript 0 implies that the derivatives are evaluated at the equilibrium orbit. n=0 gives the horizontal component of the field at the center of the magnet, which is ideally zero in the magnets considered here.

The following are defined :

$$k_n = \frac{1}{B_0\rho}\left[\frac{d^n B_y}{dx^n}\right]_0, \tag{1.5.26a}$$

$$h_n = \frac{1}{B_0\rho}\left[\frac{d^n B_x}{dx^n}\right]_0, \tag{1.5.26b}$$

with ρ as the bending radius of the particle in the magnet and $(B_0\rho)$ as the magnetic rigidity. Therefore, the Eq. (1.5.24) and Eq. (1.5.25) become

$$B_y\,(x,0) = B_0\rho\left(\frac{1}{\rho} + \sum_{n=1}^{\infty}\frac{1}{n!}k_n \;\; x^n\right), \tag{1.5.27a}$$

$$B_x\,(x,0) = B_0\rho\left(\sum_{n=0}^{\infty}\frac{1}{n!}h_n \;\; x^n\right). \tag{1.5.27b}$$

k_n and h_n used in the above equations can be related to a_n and b_n given in Eq. (1.5.19) when the horizontal and vertical components of the field are evaluated on the horizontal axis, respectively. Therefore, with $b_0 = 10^4$ and $B_{R_0} = B_0$, one obtains

$$h_n = \frac{10^{-4}\;\; n!}{\rho\;\; R_0^n}\;\; a_n, \tag{1.5.28a}$$

$$k_n = \frac{10^{-4}\;\; n!}{\rho\;\; R_0^n}\;\; b_n. \tag{1.5.28b}$$

The expressions for the horizontal and vertical component of the field in Eq. (1.5.17) can be separated out as

$$B_x = 10^{-4}B_{R_0}\sum_{n=0}^{\infty}\left[b_n \;\; sin\,(n\theta) \;\; + \;\; a_n \;\; cos\,(n\theta)\right]\left(\frac{r}{R_0}\right)^n, \tag{1.5.29a}$$

$$B_y = 10^{-4}B_{R_0}\sum_{n=0}^{\infty}\left[b_n \;\; cos\,(n\theta) \;\; - \;\; a_n \;\; sin\,(n\theta)\right]\left(\frac{r}{R_0}\right)^n. \tag{1.5.29b}$$

The radial and azimuthal components of the field can be computed by using the following relations :

$$\begin{pmatrix} B_r \\ B_\theta \end{pmatrix} = \begin{pmatrix} cos\,(\theta) & sin\,(\theta) \\ -sin\,(\theta) & cos\,(\theta) \end{pmatrix} \begin{pmatrix} B_x \\ B_y \end{pmatrix} \tag{1.5.30}$$

Therefore, the radial and azimuthal components of the field can be written as :

$$B_r = 10^{-4} B_{R_0} \sum_{n=0}^{\infty} \left[b_n \;\; sin\,[(n+1)\,\theta] \;\; + \;\; a_n \;\; cos\,[(n+1)\,\theta]\right] \left(\frac{r}{R_0}\right)^n, \tag{1.5.31a}$$

$$B_\theta = 10^{-4} B_{R_0} \sum_{n=0}^{\infty} \left[b_n \;\; cos\,[(n+1)\,\theta] \;\; - \;\; a_n \;\; sin\,[(n+1)\,\theta]\right] \left(\frac{r}{R_0}\right)^n. \tag{1.5.31b}$$

In order to represent the vector potential in terms of harmonics, the following relations can be used :

$$B_r = \frac{1}{r}\frac{\partial A_z}{\partial \theta} \quad \text{and} \quad B_\theta = -\frac{\partial A_z}{\partial r},$$

since in the 2-dimensional case $A_x = A_y = 0$. Therefore, on integrating Eqs. (1.5.31) one obtains

$$A_z = -10^{-4} B_{R_0} \sum_{n=0}^{\infty} \left(\frac{R_0}{n+1}\right) \left[b_n cos\,[(n+1)\,\theta] - a_n sin\,[(n+1)\,\theta]\right] \left(\frac{r}{R_0}\right)^{n+1}. \tag{1.5.32}$$

The inverse transform can be used to obtain individual field harmonics at a reference radius R_0 in terms of field or vector potential. For this, first a component of the field or vector potential is evaluated at a radius r and then the integration is performed over the azimuth as follows :

$$a_n = -\frac{10^4}{\pi B_{R_0}} \left(\frac{R_0}{r}\right)^n \int_0^{2\pi} B_y\,(r,\theta)\, sin\,(n\theta) \;\; d\theta, \tag{1.5.33a}$$

$$= \frac{10^4}{\pi B_{R_0}} \left(\frac{R_0}{r}\right)^n \int_0^{2\pi} B_x\,(r,\theta)\, cos\,(n\theta) \;\; d\theta, \tag{1.5.33b}$$

$$= \frac{10^4}{\pi B_{R_0}} \left(\frac{R_0}{r}\right)^n \int_0^{2\pi} B_r\,(r,\theta)\, cos\,((n+1)\,\theta) \;\; d\theta, \tag{1.5.33c}$$

$$= -\frac{10^4}{\pi B_{R_0}} \left(\frac{R_0}{r}\right)^n \int_0^{2\pi} B_\theta\,(r,\theta)\, sin\,((n+1)\,\theta) \;\; d\theta, \tag{1.5.33d}$$

$$= \frac{10^4\,(n+1)}{\pi R_0 B_{R_0}} \left(\frac{R_0}{r}\right)^{n+1} \int_0^{2\pi} A_z\,(r,\theta)\, sin\,((n+1)\,\theta) \;\; d\theta, \tag{1.5.33e}$$

$$b_n = \frac{10^4}{\pi B_{R_0}} \left(\frac{R_0}{r}\right)^n \int_0^{2\pi} B_y\,(r,\theta)\, cos\,(n\theta) \;\; d\theta, \tag{1.5.33f}$$

$$= \frac{10^4}{\pi B_{R_0}} \left(\frac{R_0}{r}\right)^n \int_0^{2\pi} B_x\,(r,\theta)\, sin\,(n\theta) \;\; d\theta, \tag{1.5.33g}$$

$$= \frac{10^4}{\pi B_{R_0}} \left(\frac{R_0}{r}\right)^n \int_0^{2\pi} B_r\,(r,\theta)\, sin\,((n+1)\,\theta) \;\; d\theta, \tag{1.5.33h}$$

$$= \frac{10^4}{\pi B_{R_0}} \left(\frac{R_0}{r}\right)^n \int_0^{2\pi} B_\theta\,(r,\theta)\, cos\,((n+1)\,\theta) \;\; d\theta, \tag{1.5.33i}$$

$$= -\frac{10^4\,(n+1)}{\pi R_0 B_{R_0}} \left(\frac{R_0}{r}\right)^{n+1} \int_0^{2\pi} A_z\,(r,\theta)\, cos\,((n+1)\,\theta) \;\; d\theta. \tag{1.5.33j}$$

For the primary harmonic component $n = m$, when the field is perpendicular to the horizontal plane, one obtains

$$b_m = 10^4 \quad \text{and} \quad a_m = 0.$$

1.5.3. Analytic Expressions for Accelerator Magnets

Analytic expressions for the basic cosine theta superconducting magnet design have been previously obtained and described by several authors [12-18,144,175,179]. Superconducting accelerator magnets are usually long cylindrical magnets with the current flowing parallel to the magnet axis (z-axis). The geometry of these magnets is such that one can compute the field in the body of the magnet by assuming that the current is carried by a large number of wires parallel to the z-axis. The total field is obtained by simply superimposing the field created by these wires. For this purpose, it is suitable to carry out a 2-dimensional analysis in the cylindrical coordinate system. A three dimensional analysis will be necessary for computing the field at the ends of the magnet.

Accelerator magnets are designed to produce a well defined field in the aperture of the magnets. The *field* in the aperture is constant for *dipoles*, the *first derivative* of the field is constant for *quadrupoles* and, in general, the n^{th} *derivative* is constant for the n^{th}-order *multipole*. In the following sections, the current distributions needed to produce such multipole fields will be obtained.

1.5.3.1. Field and Vector Potential due to a Line Current

To compute the magnetic field and vector potential due to a single infinitely long wire, it is assumed to carry a current I in the z-direction which is perpendicular to the plane of paper. The field outside this wire at a perpendicular distance R from it will be computed. The cylindrical coordinate system is used to take advantage of the symmetry of the problem.

The magnetic field produced by this wire can be directly calculated by using the integral equation $\oint \vec{H} \cdot ds = I$ (Eqs. (1.5.9)) which gives:

$$H = \frac{I}{2\pi R}, \tag{1.5.34}$$

and in a medium having a relative permeability of μ

$$B = \frac{I\mu\mu_0}{2\pi R}. \tag{1.5.35}$$

The components of vector potential in cylindrical and Cartesian geometry can be written as

$$A_z = \frac{\mu\mu_0 I}{2\pi} ln\left(\frac{1}{R}\right), \tag{1.5.36a}$$

$$A_r = A_\theta = 0, \tag{1.5.36b}$$

$$A_x = A_y = 0. \tag{1.5.36c}$$

The validity of the above relation is verified when the curl of the vector potential is taken to obtain the magnetic field as per Eqs. (1.5.7). This gives $B_r = B_z = 0$ and $B_\theta = \frac{\mu\mu_0 I}{2\pi R}$; which is the same as in Eqs. (1.5.34) with only one component of the field present.

In accelerator magnets, the magnetic field and vector potential are usually expressed in terms of harmonic components. To develop this formalism a line current is assumed to be located at a point "Q" (at $\vec{a}$) and the magnetic field produced by it is computed at point "P" (at $\vec{r}$), as shown in Fig. 1.5.1. The distance between the two is $\vec{R} = \vec{r} - \vec{a}$ with the magnitude $|R| = \sqrt{r^2 + a^2 - 2racos(\theta - \phi)}$.

In this section, the computations will be mostly done in a space free of magnetic material where the relative permeability μ is one. Moreover, to simplify the expressions to follow, Eq. (1.5.36a) is re-written after adding a constant :

$$A_z\,(r,\theta) = -\frac{\mu_o I}{2\pi} ln\left(\frac{R}{a}\right); \tag{1.5.37}$$

the addition of such a constant does not change the magnetic field which is a derivative of A_z.

Now $A_z(r,\theta)$ will be given in terms of a series expansion containing, in general, summation of terms like $(\frac{r}{a})^m$ and $(\frac{a}{r})^m$, together with trigonometric functions like $cos(m\theta)$ and $sin(m\theta)$. The exact solution will depend on a particular problem. For example, in the solution of the case when r approaches the origin ($r \to 0$), the $(\frac{a}{r})^m$ terms can't be present. Similarly in the solution of the case when r approaches infinity ($r \to \infty$), the $(\frac{a}{r})^m$ terms can't be present.

In order to obtain an expansion of the ln in Eq. (1.5.37), the following manipulation is carried out :

$$\begin{aligned}
R^2 &= r^2 + a^2 - 2ra\;\; cos\,(\theta - \phi)\,, \\
\frac{R}{a} &= \left(1 - \left(\frac{r}{a}\right)e^{i(\phi-\theta)}\right)^{\frac{1}{2}} \cdot \left(1 - \left(\frac{r}{a}\right)e^{-i(\phi-\theta)}\right)^{\frac{1}{2}}, \\
ln\left(\frac{R}{a}\right) &= \frac{1}{2} ln\left(1 - \left(\frac{r}{a}\right)e^{i(\phi-\theta)}\right) + \frac{1}{2} ln\left(1 - \left(\frac{r}{a}\right)e^{-i(\phi-\theta)}\right).
\end{aligned}$$

For $|z| < 1$, the logarithmic expansion is given by

$$ln\,(1-z) = -\left[z + \left(\frac{z^2}{2}\right) + \left(\frac{z^3}{3}\right) + ...\right] = -\sum_{n=1}^{\infty}\frac{z^n}{n}.$$

Therefore, for r < a

$$ln\left(\frac{R}{a}\right) = -\left[\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)\left(\frac{r}{a}\right)^n e^{i\;n(\phi-\theta)} + \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)\left(\frac{r}{a}\right)^n e^{-i\;n(\phi-\theta)}\right],$$

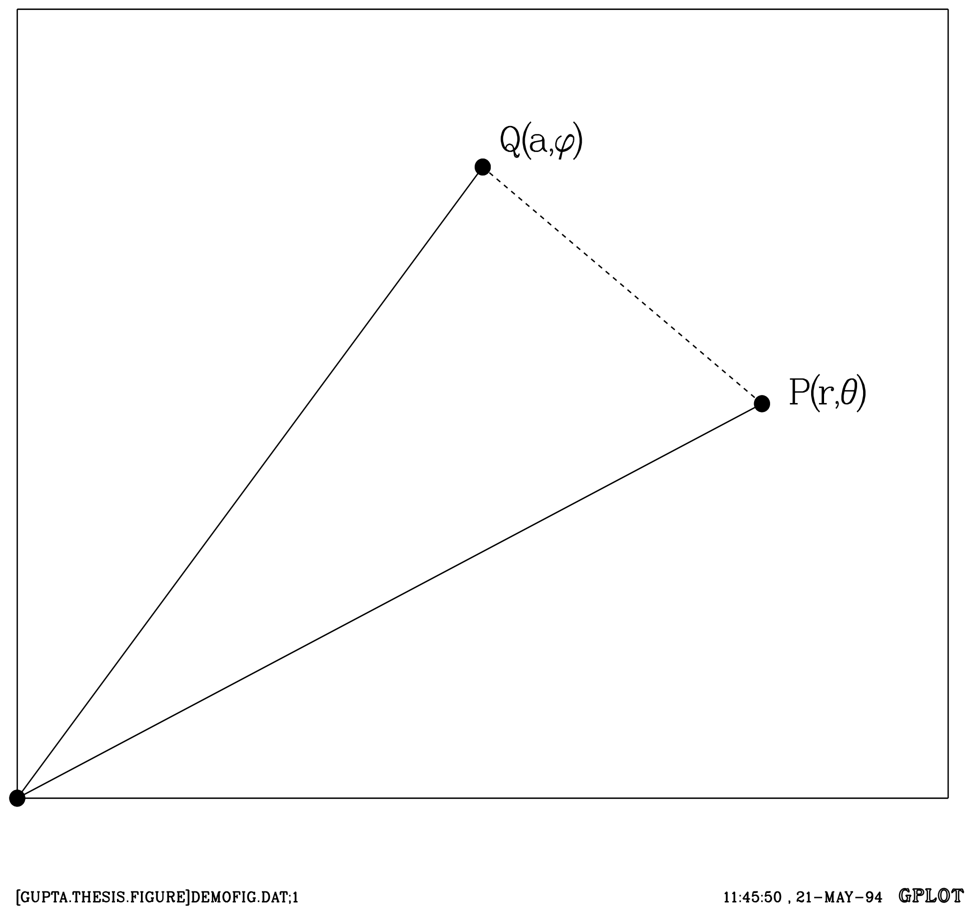

Figure 1.5.1: Computation of the field at a location "P" produced by the line current located at a position "Q".

$$ln\left(\frac{R}{a}\right) = -\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)\left(\frac{r}{a}\right)^{n} cos\left(n\left(\phi-\theta\right)\right). \tag{1.5.38}$$

Substituting Eqs. (1.5.38) in Eqs. (1.5.37) the desired expansion for the vector potential is obtained (for r < a) :

$$A_z\left(r,\theta\right) = \frac{\mu_o I}{2\pi}\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)\left(\frac{r}{a}\right)^{n} cos\left(n\left(\phi-\theta\right)\right). \tag{1.5.39}$$

The magnetic field components are obtained by using Eqs. (1.5.7) and Eqs. (1.5.37) with $A_r = A_\theta = 0$:

$$B_r = \frac{1}{r}\left(\frac{\partial A_z}{\partial \theta}\right), \tag{1.5.40a}$$

$$B_\theta = -\frac{\partial A_z}{\partial r}, \tag{1.5.40b}$$

$$B_z = 0. \tag{1.5.40c}$$

Therefore, for r $<$ a, one would obtain :

$$B_r = \frac{\mu_o I}{2\pi a}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^{n-1} sin\left[(n)\left(\phi-\theta\right)\right], \tag{1.5.41a}$$

$$B_\theta = -\frac{\mu_o I}{2\pi a}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^{n-1} cos\left[(n)\left(\phi-\theta\right)\right], \tag{1.5.41b}$$

$$B_z = 0. \tag{1.5.41c}$$

In order to compute the harmonics components, the above equations are compared with Eqs. (1.5.31). It should be noted that there the summation starts from $n = 0$ instead of $n = 1$ in Eq. (1.5.39). The following expressions for the normal and skew harmonics at a reference radius R_0 are obtained for a line current located at (a, ϕ) :

$$b_n = 10^4\left(\frac{R_0}{a}\right)^n \; cos\left[(n+1)\,\phi\right], \tag{1.5.42a}$$

$$a_n = -10^4\left(\frac{R_0}{a}\right)^n \; sin\left[(n+1)\,\phi\right], \tag{1.5.42b}$$

and $B_{R_o} = -\frac{\mu_o I}{2\pi a}$.

For r $>$ a case, the following rearrangement is performed to obtain an appropriate expansion :

$$\frac{R}{a} = \left(\frac{r}{a}\right)\left(1-\left(\frac{a}{r}\right)e^{i(\phi-\theta)}\right)^{\frac{1}{2}}\cdot\left(1-\left(\frac{a}{r}\right)e^{-i(\phi-\theta)}\right)^{\frac{1}{2}},$$

$$ln\left(\frac{R}{a}\right) = ln\left(\frac{r}{a}\right)+\left[\frac{1}{2}ln\left(1-\left(\frac{a}{r}\right)e^{i(\phi-\theta)}\right)+\frac{1}{2}ln\left(1-\left(\frac{a}{r}\right)e^{-i(\phi-\theta)}\right)\right],$$

$$ln\left(\frac{R}{a}\right) = ln\left(\frac{r}{a}\right)-\left[\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)\left(\frac{a}{r}\right)^n e^{i\;n(\phi-\theta)}+\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)\left(\frac{a}{r}\right)^n e^{-i\;n(\phi-\theta)}\right],$$

$$ln\left(\frac{R}{a}\right) = ln\left(\frac{r}{a}\right)-\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)\left(\frac{a}{r}\right)^n cos\left(n\left(\phi-\theta\right)\right). \tag{1.5.43}$$

Therefore, for r $>$ a, one obtains the following expression for the vector potential :

$$A_z\left(r,\theta\right) = -\frac{\mu_o I}{2\pi}ln\left(\frac{r}{a}\right)+\frac{\mu_o I}{2\pi}\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)\left(\frac{a}{r}\right)^n cos\left(n\left(\phi-\theta\right)\right). \tag{1.5.44}$$

The magnetic field components are obtained by using Eqs. (1.5.40) :

$$B_r = \frac{\mu_o I}{2\pi a}\sum_{n=0}^{\infty}\left(\frac{a}{r}\right)^{n+1} sin\left(n\left(\phi-\theta\right)\right), \tag{1.5.45a}$$

$$B_\theta = \frac{\mu_o I}{2\pi a}\sum_{n=0}^{\infty}\left(\frac{a}{r}\right)^{n+1} cos\left(n\left(\phi-\theta\right)\right), \tag{1.5.45b}$$

$$B_z = 0. \tag{1.5.45c}$$

It may be noted that in the expression for B_θ, the summation in n starts from $n = 0$ instead of $n = 1$. The (B_x, B_y) components of the field can be computed using the following relation:

$$\begin{pmatrix} B_x \\ B_y \end{pmatrix} = \begin{pmatrix} cos\left(\theta\right) & -sin\left(\theta\right) \\ sin\left(\theta\right) & cos\left(\theta\right) \end{pmatrix} \begin{pmatrix} B_r \\ B_\theta \end{pmatrix}. \tag{1.5.46}$$

1.5.3.2. Line Current in a Cylindrical Iron Cavity

Expressions are obtained here for the vector potential and magnetic field due to an infinitely long paraxial filament of current I at a radius a in a cylindrical cavity having a radius $R_f > a$. The iron is infinitely long and infinitely thick and has a constant relative permeability μ, which is referred to here simply as permeability following the convention explained earlier. The method of image currents can be applied to include the contribution from the iron [179]. The expressions are obtained here by matching the boundary conditions at the interface of the air and iron boundary [19]. General expressions for the vector potential and the components of the field in the region $a < r < R_f$, are given by :

$$A_z(r,\theta) = -\frac{\mu_o I}{2\pi} ln\left(\frac{r}{a}\right) + \frac{\mu_o I}{2\pi}\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)\left(\frac{a}{r}\right)^n cos\left(n\left(\phi-\theta\right)\right)$$
$$+ \mu_o \sum_{n=1}^{\infty} E_n r^n cos\left(n\left(\phi-\theta\right)\right), \tag{1.5.47}$$

$$B_r = \frac{\mu_o I}{2\pi a}\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^{n+1} sin\left(n\left(\phi-\theta\right)\right)$$
$$+ \mu_o \sum_{n=1}^{\infty} n E_n r^{n-1} sin\left(n\left(\phi-\theta\right)\right), \tag{1.5.48a}$$

$$H_\theta = \frac{I}{2\pi a}\sum_{n=0}^{\infty}\left(\frac{a}{r}\right)^{n+1} cos\left(n\left(\phi-\theta\right)\right)$$
$$- \sum_{n=1}^{\infty} n E_n r^{n-1} cos\left(n\left(\phi-\theta\right)\right), \tag{1.5.48b}$$

and in the region $r > R_f$:

$$A_z(r,\theta) = \mu\mu_o F_o ln\left(\frac{r}{a}\right)$$
$$+ \mu\mu_o \sum_{n=1}^{\infty} F_n \left(\frac{1}{r}\right)^n cos\left(n\left(\phi-\theta\right)\right), \tag{1.5.49}$$

$$B_r = \mu\mu_o \sum_{n=1}^{\infty} n F_n \left(\frac{1}{r}\right)^{n+1} sin\left(n\left(\phi-\theta\right)\right), \tag{1.5.50a}$$

$$H_\theta = -\frac{F_o}{r}$$
$$+ \sum_{n=1}^{\infty} n F_n \left(\frac{1}{r}\right)^{n+1} cos\left(n\left(\phi-\theta\right)\right), \tag{1.5.50b}$$

where E_n and F_n are coefficients which can be determined by the boundary conditions at $r = R_f$ that

$$(B_r)_{air} = (B_r)_{iron} ,$$
$$(H_\theta)_{air} = (H_\theta)_{iron} ,$$

i.e. the normal component of B and the azimuthal component of H are continuous. Therefore, the required boundary conditions at $r = R_f$ for $n \neq 0$ gives :

$$\frac{\mu_o I}{2\pi a}\left(\frac{a}{R_f}\right)^{n+1} + n\ \mu_o\ E_n {R_f}^{n-1} = n\ \mu_0\ \mu\ F_n \left(\frac{1}{R_f}\right)^{n+1},$$
$$\frac{I}{2\pi a}\left(\frac{a}{R_f}\right)^{n+1} - \ n\ E_n {R_f}^{n-1} = n\ F_n \left(\frac{1}{R_f}\right)^{n+1},$$

which gives

$$E_n = \frac{1}{n}\frac{\mu-1}{\mu+1}\ \frac{I}{2\pi}\ \left(\frac{a}{R_f^2}\right)^n, \tag{1.5.51a}$$
$$F_n = \frac{1}{n}\frac{2}{\mu+1}\ \frac{I}{2\pi a}\ a^{n+1}. \tag{1.5.51b}$$

The $n = 0$ term appears only in the expression for H_θ and on matching the boundary condition, one obtains :

$$\frac{I}{2\pi R_f} = -\frac{F_o}{R_f},$$

which gives

$$F_o = -\frac{I}{2\pi}. \tag{1.5.52}$$

The expressions for vector potential and field components for $a < r < R_f$ case are obtained when E_n from Eq. (1.5.51a) is substituted in Eq. (1.5.47) and Eqs. (1.5.48) :

$$\begin{aligned} A_z(r,\theta) = &-\frac{\mu_o I}{2\pi} ln\left(\frac{r}{a}\right) \\ &+ \frac{\mu_o I}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{a}{r}\right)^n cos\left(n\left(\phi-\theta\right)\right)\left[1+\frac{\mu-1}{\mu+1}\left(\frac{r}{R_f}\right)^{2n}\right], \end{aligned} \tag{1.5.53}$$
$$B_r = \frac{\mu_o I}{2\pi a}\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^{n+1} sin\left(n\left(\phi-\theta\right)\right)\left[1+\frac{\mu-1}{\mu+1}\left(\frac{r}{R_f}\right)^{2n}\right], \tag{1.5.54a}$$
$$B_\theta = \frac{\mu_o I}{2\pi r} + \frac{\mu_o I}{2\pi a}\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^{n+1} cos\left(n\left(\phi-\theta\right)\right)\left[1-\frac{\mu-1}{\mu+1}\left(\frac{r}{R_f}\right)^{2n}\right]. \tag{1.5.54b}$$

In the above equations, the second term in the square brackets is the additional contribution of the iron to the field produced by the coil.

To obtain the expressions for the vector potential and field for $r < a$ it must be noted that a current filament is present at $r = a$. However, the radial component of the field B_r must still be continuous, i.e. at $r = a$

$$B_r\,(in) = B_r\,(out),$$

where $B_r(in)$ and $B_r(out)$ are the magnetic induction for $r < a$ and $a < r < R_f$ respectively. The presence of the source (current), however, gives a discontinuity in the azimuthal component of the field H_θ with $H_\theta(in)$ - $H_\theta(out)$ determined by the current density at $r = a$. A general expression for the vector potential for $r < a$ is given by (see Eq. (1.5.39)) :

$$A_z\left(r,\theta\right)=\mu_o\sum_{n=1}^{\infty}I_n\;\;r^n cos\left(n\left(\phi-\theta\right)\right), \tag{1.5.55}$$

where the I_n are unknown coefficients. Using Eqs. (1.5.40) :

$$B_r=\mu_o\sum_{n=1}^{\infty}I_n\;\;n\;\;r^{n-1}sin\left(n\left(\phi-\theta\right)\right), \tag{1.5.56a}$$

$$B_\theta=-\mu_o\sum_{n=1}^{\infty}I_n\;\;n\;\;r^{n-1}cos\left(n\left(\phi-\theta\right)\right), \tag{1.5.56b}$$

$$B_z=0. \tag{1.5.56c}$$

In order for B_r to be continuous at $r = a$ one obtains from Eq. (1.5.56a) and Eq. (1.5.54a) :

$$I_n=\frac{I}{2\pi}\left(\frac{1}{n}\right)\left(\frac{1}{a}\right)^n\left[1+\frac{\mu-1}{\mu+1}\left(\frac{a}{R_f}\right)^{2n}\right]. \tag{1.5.57}$$

Using this in Eqs. (1.5.56) gives the expressions for the field and vector potential for $r < a$ as :

$$A_z\left(r,\theta\right)=\frac{\mu_o I}{2\pi}\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)\left(\frac{r}{a}\right)^n cos\left(n\left(\phi-\theta\right)\right)\left[1+\frac{\mu-1}{\mu+1}\left(\frac{a}{R_f}\right)^{2n}\right]. \tag{1.5.58}$$

$$B_r=\frac{\mu_o I}{2\pi a}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^{n-1} sin\left(n\left(\phi-\theta\right)\right)\left[1+\frac{\mu-1}{\mu+1}\left(\frac{a}{R_f}\right)^{2n}\right], \tag{1.5.59a}$$

$$B_\theta=-\frac{\mu_o I}{2\pi a}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^{n-1} cos\left(n\left(\phi-\theta\right)\right)\left[1+\frac{\mu-1}{\mu+1}\left(\frac{a}{R_f}\right)^{2n}\right], \tag{1.5.59b}$$

$$B_z=0. \tag{1.5.59c}$$

To compute the field harmonics the procedure of Eqs. (1.5.42) is repeated. As before, the summation over n in the above is now changed so that it starts from n=0 instead of n=1.

$$b_n=10^4\left(\frac{R_0}{a}\right)^n\;\;cos\left(\left(n+1\right)\phi\right)\left[1+\frac{\mu-1}{\mu+1}\left(\frac{a}{R_f}\right)^{2(n+1)}\right], \tag{1.5.60a}$$

$$a_n=-10^4\left(\frac{R_0}{a}\right)^n\;\;sin\left(\left(n+1\right)\phi\right)\left[1+\frac{\mu-1}{\mu+1}\left(\frac{a}{R_f}\right)^{2(n+1)}\right]. \tag{1.5.60b}$$

All expressions derived so far reproduce the results obtained from the method of images [179] which says that the effect of iron can be replaced by an additional line current $I^{'}$ located at $(a^{'}, \phi)$ with

$$I^{'} = \left(\frac{\mu - 1}{\mu + 1}\right) I,$$
$$a^{'} = \frac{R_f^2}{a}.$$

The expressions for vector potential and field components for $r > R_f$ case are obtained when F_n from Eq. (1.5.51b) and Eq. (1.5.52) are substituted in Eqs. (1.5.49) and Eqs. (1.5.50) :

$$A_z\left(r, \theta\right) = -\frac{\mu\mu_o I}{2\pi} ln\left(\frac{r}{a}\right) + \frac{2\mu}{\mu + 1}\frac{\mu_o I}{2\pi}\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)\left(\frac{a}{r}\right)^n cos\left(n\left(\phi - \theta\right)\right) \qquad (1.5.61)$$

$$B_r = \frac{2\mu}{\mu + 1}\frac{\mu_o I}{2\pi a}\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^{n+1} sin\left(n\left(\phi - \theta\right)\right), \qquad (1.5.62a)$$

$$B_\theta = \frac{\mu\mu_o I}{2\pi r} + \frac{2\mu}{\mu + 1}\frac{\mu_o I}{2\pi a}\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^{n+1} cos\left(n\left(\phi - \theta\right)\right). \qquad (1.5.62b)$$

1.5.3.3. Line Current in a Cylindrical Iron Shell

In deriving the expressions for the vector potential and field due to a line current inside an cylindrical iron it was assumed in the last section that the iron outer boundary extends to infinity. This is, however, not the case in practice. If the outer diameter of the cylindrical iron shell is R_a, then the general expressions for the vector potential in the various regions are given by :

$$A_z(r,\theta) = \mu_o \sum_{n=1}^{\infty} I_n^{'} \; r^n cos\left(n\left(\phi-\theta\right)\right), \qquad [for \quad \mathrm{r < a}] \qquad (1.5.63a)$$

$$A_z(r,\theta) = -\frac{\mu_o I}{2\pi} ln\left(\frac{r}{a}\right) + \frac{\mu_o I}{2\pi}\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)\left(\frac{a}{r}\right)^n cos\left(n\left(\phi-\theta\right)\right)$$
$$+ \mu_o \sum_{n=1}^{\infty} E_n^{'} r^n cos\left(n\left(\phi-\theta\right)\right), \qquad [for \quad \mathrm{a < r < R_f}] \qquad (1.5.63b)$$

$$A_z(r,\theta) = \mu\mu_o F_o^{'} ln\left(\frac{r}{a}\right)$$
$$+ \mu\mu_o \sum_{n=1}^{\infty} F_n^{'}\left(\frac{1}{r}\right)^n cos\left(n\left(\phi-\theta\right)\right)$$
$$+ \mu\mu_o \sum_{n=1}^{\infty} G_n^{'} r^n cos\left(n\left(\phi-\theta\right)\right), \qquad [for \quad \mathrm{R_f < r < R_a}] \qquad (1.5.63c)$$

$$A_z(r,\theta) = \mu_o H_o^{'} ln\left(\frac{r}{a}\right)$$
$$+ \mu_o \sum_{n=1}^{\infty} H_n^{'}\left(\frac{1}{r}\right)^n cos\left(n\left(\phi-\theta\right)\right), \qquad [for \quad \mathrm{r > R_a}] \qquad (1.5.63d)$$

Following an approach similar to one used in previous section, the five coefficients $(E_n^{'},\ F_n^{'},\ G_n^{'},\ H_n^{'},\ I_n^{'})$ are obtained by matching the five boundary conditions (B_r is continuous at $r = a$, $r = R_f$ and $r = R_a$ and B_θ is continuous at $r = R_f$ and $r = R_a$). The results of that exercise for $n > 0$ are given here :

$$I_n^{'} = \frac{I}{2\pi}\left(\frac{1}{n}\right)\left(\frac{1}{a}\right)^n \left[1 + \frac{\mu-1}{\mu+1}\left(\frac{a}{R_f}\right)^{2n} \frac{\left[1-\left(\frac{R_f}{R_a}\right)^{2n}\right]}{\left[1-\left(\frac{\mu-1}{\mu+1}\right)^2\left(\frac{R_f}{R_a}\right)^{2n}\right]}\right] \quad , \qquad (1.5.64a)$$

$$E_n^{'} = \frac{1}{n}\frac{\mu-1}{\mu+1}\,\frac{I}{2\pi}\left(\frac{a}{R_f^2}\right)^n \frac{\left[1-\left(\frac{R_f}{R_a}\right)^{2n}\right]}{\left[1-\left(\frac{\mu-1}{\mu+1}\right)^2\left(\frac{R_f}{R_a}\right)^{2n}\right]} \quad , \qquad (1.5.64b)$$

$$F_n^{'} = \frac{1}{\mu+1}\,\frac{I}{n\pi}\,\frac{a^n}{\left[1-\left(\frac{\mu-1}{\mu+1}\right)^2\left(\frac{R_f}{R_a}\right)^{2n}\right]} \quad , \qquad (1.5.64c)$$

$$G_n^{'} = -\frac{(\mu-1)}{(\mu+1)^2}\ \frac{I}{n\pi}\frac{\left(\frac{a}{R_a^2}\right)^n}{\left[1-\left(\frac{\mu-1}{\mu+1}\right)^2\left(\frac{R_f}{R_a}\right)^{2n}\right]} \quad , \tag{1.5.64d}$$

$$H_n^{'} = \frac{2\mu}{(\mu+1)^2}\ \frac{I}{n\pi}\ \frac{a^n}{\left[1-\left(\frac{\mu-1}{\mu+1}\right)^2\left(\frac{R_f}{R_a}\right)^{2n}\right]} \quad , \tag{1.5.64e}$$

and for $n = 0$, the terms are:

$$F_o^{'} \ = \ H_o^{'} = -\frac{I}{2\pi}. \tag{1.5.65}$$

Therefore, the expressions for the vector potential and field components in various regions due to a line current I at (a,θ) inside a cylindrical iron shell having inner radius R_f and outer radius R_a are given as follows (in each case $B_z(r,\theta)=0$) :

Inside Coil(r < a)

$$A_z(r,\theta) = \frac{\mu_o I}{2\pi}\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)\left[1+\frac{\mu-1}{\mu+1}\left(\frac{a}{R_f}\right)^{2n}\frac{\left[1-\left(\frac{R_f}{R_a}\right)^{2n}\right]}{\left[1-\left(\frac{\mu-1}{\mu+1}\right)^2\left(\frac{R_f}{R_a}\right)^{2n}\right]}\right]\times$$
$$\left(\frac{r}{a}\right)^n cos\left(n\left(\phi-\theta\right)\right) \quad . \tag{1.5.66}$$

$$B_r = \frac{\mu_o I}{2\pi a}\sum_{n=1}^{\infty}\left[1+\frac{\mu-1}{\mu+1}\left(\frac{a}{R_f}\right)^{2n}\frac{\left[1-\left(\frac{R_f}{R_a}\right)^{2n}\right]}{\left[1-\left(\frac{\mu-1}{\mu+1}\right)^2\left(\frac{R_f}{R_a}\right)^{2n}\right]}\right]\times$$
$$\left(\frac{r}{a}\right)^{n-1} sin\left(n\left(\phi-\theta\right)\right) \quad , \tag{1.5.67a}$$

$$B_\theta = -\frac{\mu_o I}{2\pi a}\sum_{n=1}^{\infty}\left[1+\frac{\mu-1}{\mu+1}\left(\frac{a}{R_f}\right)^{2n}\frac{\left[1-\left(\frac{R_f}{R_a}\right)^{2n}\right]}{\left[1-\left(\frac{\mu-1}{\mu+1}\right)^2\left(\frac{R_f}{R_a}\right)^{2n}\right]}\right]\times$$
$$\left(\frac{r}{a}\right)^{n-1} cos\left(n\left(\phi-\theta\right)\right) \quad , \tag{1.5.67b}$$

Between Coil and Iron ($\mathbf{a < r < R_f}$)

$$A_z(r,\theta) = -\frac{\mu_o I}{2\pi}ln\left(\frac{r}{a}\right)+\frac{\mu_o I}{2\pi}\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)\left(1+\frac{\frac{\mu-1}{\mu+1}\left(\frac{r}{R_f}\right)^{2n}\left[1-\left(\frac{R_f}{R_a}\right)^{2m}\right]}{1-\left(\frac{\mu-1}{\mu+1}\right)^2\left(\frac{R_f}{R_a}\right)^{2n}}\right)\times$$
$$\left(\frac{a}{r}\right)^n cos\left(n\left(\phi-\theta\right)\right) \ , \tag{1.5.68}$$

$$B_r = \frac{\mu_o I}{2\pi a}\sum_{n=1}^{\infty}\left(1+\frac{\frac{\mu-1}{\mu+1}\left(\frac{r}{R_f}\right)^{2n}\left[1-\left(\frac{R_f}{R_a}\right)^{2m}\right]}{1-\left(\frac{\mu-1}{\mu+1}\right)^2\left(\frac{R_f}{R_a}\right)^{2n}}\right)\times$$

$$\left(\frac{a}{r}\right)^{n+1} sin\left(n\left(\phi-\theta\right)\right) \quad , \tag{1.5.69a}$$

$$B_\theta = \frac{\mu_o I}{2\pi r} + \frac{\mu_o I}{2\pi a}\sum_{n=1}^{\infty}\left(1-\frac{\frac{\mu-1}{\mu+1}\left(\frac{r}{R_f}\right)^{2n}\left[1-\left(\frac{R_f}{R_a}\right)^{2m}\right]}{1-\left(\frac{\mu-1}{\mu+1}\right)^2\left(\frac{R_f}{R_a}\right)^{2n}}\right)\times$$

$$\left(\frac{a}{r}\right)^{n+1} cos\left(n\left(\phi-\theta\right)\right) \quad . \tag{1.5.69b}$$

<u>**Inside Iron ($\mathbf{R_f < r < R_a}$)**</u>

$$A_z\left(r,\theta\right) = -\frac{\mu\mu_o I}{2\pi} ln\left(\frac{r}{a}\right)$$

$$+\frac{\mu\mu_o I}{\pi\left(\mu+1\right)}\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)\ \frac{1-\frac{\mu-1}{\mu+1}\left(\frac{r}{R_a}\right)^{2n}}{1-\left(\frac{\mu-1}{\mu+1}\right)^2\left(\frac{R_f}{R_a}\right)^{2n}} \quad \times$$

$$\left(\frac{a}{r}\right)^{n} cos\left(n\left(\phi-\theta\right)\right) \quad , \tag{1.5.70}$$

$$B_r = \frac{\mu\mu_o I}{\pi a\left(\mu+1\right)}\sum_{n=1}^{\infty}\frac{1-\frac{\mu-1}{\mu+1}\left(\frac{r}{R_a}\right)^{2n}}{1-\left(\frac{\mu-1}{\mu+1}\right)^2\left(\frac{R_f}{R_a}\right)^{2n}} \quad \times$$

$$\left(\frac{a}{r}\right)^{n+1} sin\left(n\left(\phi-\theta\right)\right) \quad , \tag{1.5.71a}$$

$$B_\theta = \frac{\mu\mu_o I}{2\pi r} + \frac{\mu\mu_o I}{\pi a\left(\mu+1\right)}\sum_{n=1}^{\infty}\frac{1+\frac{\mu-1}{\mu+1}\left(\frac{r}{R_a}\right)^{2n}}{1-\left(\frac{\mu-1}{\mu+1}\right)^2\left(\frac{R_f}{R_a}\right)^{2n}} \quad \times$$

$$\left(\frac{a}{r}\right)^{n+1} cos\left(n\left(\phi-\theta\right)\right) \quad . \tag{1.5.71b}$$

<u>**Outside Iron ($\mathbf{r > R_a}$)**</u>

$$A_z\left(r,\theta\right) = -\frac{\mu_o I}{2\pi} ln\left(\frac{r}{a}\right) + \frac{2\mu\mu_o I}{\pi\left(\mu+1\right)^2}\sum_{n=1}^{\infty}\ \frac{1}{1-\left(\frac{\mu-1}{\mu+1}\right)^2\left(\frac{R_f}{R_a}\right)^{2n}} \ \times$$

$$\left(\frac{1}{n}\right)\left(\frac{a}{r}\right)^{n} cos\left(n\left(\phi-\theta\right)\right) \tag{1.5.72}$$

$$B_r = \frac{2\mu}{\left(\mu+1\right)^2}\frac{\mu_o I}{\pi a}\sum_{n=1}^{\infty}\ \frac{1}{1-\left(\frac{\mu-1}{\mu+1}\right)^2\left(\frac{R_f}{R_a}\right)^{2n}} \ \times$$

$$\left(\frac{a}{r}\right)^{n+1} sin\left(n\left(\phi-\theta\right)\right), \tag{1.5.73a}$$

$$B_\theta = \frac{\mu_o I}{2\pi r} + \frac{2\mu}{\left(\mu+1\right)^2}\frac{\mu_o I}{\pi a}\sum_{n=0}^{\infty}\ \frac{1}{1-\left(\frac{\mu-1}{\mu+1}\right)^2\left(\frac{R_f}{R_a}\right)^{2n}} \ \times$$

$$\left(\frac{a}{r}\right)^{n+1} cos\left(n\left(\phi-\theta\right)\right). \tag{1.5.73b}$$

Field Harmonics

The field harmonics are given by :

$$b_n = 10^4 \left(\frac{R_0}{a}\right)^n \quad cos\left((n+1)\,\phi\right) \left[1 + \frac{\mu-1}{\mu+1}\left(\frac{a}{R_f}\right)^{2(n+1)} \frac{1-\left(\frac{R_f}{R_a}\right)^{2(n+1)}}{1-\left(\frac{\mu-1}{\mu+1}\right)^2\left(\frac{R_f}{R_a}\right)^{2(n+1)}}\right], \tag{1.5.74a}$$

$$a_n = -10^4 \left(\frac{R_0}{a}\right)^n \quad sin\left((n+1)\,\phi\right) \left[1 + \frac{\mu-1}{\mu+1}\left(\frac{a}{R_f}\right)^{2(n+1)} \frac{1-\left(\frac{R_f}{R_a}\right)^{2(n+1)}}{1-\left(\frac{\mu-1}{\mu+1}\right)^2\left(\frac{R_f}{R_a}\right)^{2(n+1)}}\right]. \tag{1.5.74b}$$

1.5.3.4. Field and Harmonics due to Current Blocks in Air

The expressions derived for the line current in the section 1.5.3.1 are extended here for one or more blocks of current. The geometry of the problem is such that a wire is replaced by a radial block between radii ρ_1 and ρ_2 and angle ϕ_1 and ϕ_2. The block has a constant current density J such that the total current is still I with $I = \frac{1}{2}J(\rho_2^2 - \rho_1^2)(\phi_2 - \phi_1)$. To compute the vector potential and component of field at (r, θ) Eq. (1.5.36) and Eqs. (1.5.41) should be integrated [179] as (for $r < \rho_1$) :

$$A_z(r,\theta) = \sum_{n=1}^{\infty} \int_{\rho_1}^{\rho_2} \frac{\mu_o J}{2\pi}\left(\frac{1}{n}\right)\left(\frac{r}{a}\right)^n a \; da \int_{\phi_1}^{\phi_2} cos\left[n\left(\phi - \theta\right)\right] \; d\phi, \tag{1.5.75}$$

$$B_r(r,\theta) = \sum_{n=1}^{\infty} \int_{\rho_1}^{\rho_2} \frac{\mu_o J}{2\pi a}\left(\frac{r}{a}\right)^{n-1} a \; da \int_{\phi_1}^{\phi_2} sin\left[n\left(\phi - \theta\right)\right] d\phi, \tag{1.5.76a}$$

$$B_\theta(r,\theta) = -\sum_{n=1}^{\infty} \int_{\rho_1}^{\rho_2} \frac{\mu_o J}{2\pi a}\left(\frac{r}{a}\right)^{n-1} a \; da \int_{\phi_1}^{\phi_2} cos\left[n\left(\phi - \theta\right)\right] \; d\phi. \tag{1.5.76b}$$

The integration of the above equations for the vector potential and the field components gives :

$$\begin{aligned} A_z(r,\theta) = {} & \frac{\mu_o J r}{2\pi}(\rho_2 - \rho_1)\left[sin\left(\phi_2 - \theta\right) - sin\left(\phi_1 - \theta\right)\right] \\ & + \frac{\mu_o J r^2}{8\pi} ln\left(\frac{\rho_2}{\rho_1}\right)\left[sin\left(2\left(\phi_2 - \theta\right)\right) - sin\left(2\left(\phi_1 - \theta\right)\right)\right] \\ & - \frac{\mu_o J}{2\pi}\sum_{n=3}^{\infty}\frac{r^n}{n^2\left(n-2\right)}\left(\frac{1}{\rho_2^{n-2}} - \frac{1}{\rho_1^{n-2}}\right) \times \\ & \left[sin\left(n\left(\phi_2 - \theta\right)\right) - sin\left(n\left(\phi_1 - \theta\right)\right)\right], \end{aligned} \tag{1.5.77}$$

$$\begin{aligned} B_r(r,\theta) = {} & -\frac{\mu_o J}{2\pi}(\rho_2 - \rho_1)\left[cos\left(\phi_2 - \theta\right) - cos\left(\phi_1 - \theta\right)\right] \\ & - \frac{\mu_o J r}{4\pi} ln\left(\frac{\rho_2}{\rho_1}\right)\left[cos\left(2\left(\phi_2 - \theta\right)\right) - cos\left(2\left(\phi_1 - \theta\right)\right)\right] \\ & + \frac{\mu_o J}{2\pi}\sum_{n=3}^{\infty}\frac{r^{n-1}}{n\left(n-2\right)}\left(\frac{1}{\rho_2^{n-2}} - \frac{1}{\rho_1^{n-2}}\right) \times \\ & \left[cos\left(n\left(\phi_2 - \theta\right)\right) - cos\left(n\left(\phi_1 - \theta\right)\right)\right], \end{aligned} \tag{1.5.78a}$$

$$\begin{aligned} B_\theta(r,\theta) = {} & -\frac{\mu_o J}{2\pi}(\rho_2 - \rho_1)\left[sin\left(\phi_2 - \theta\right) - sin\left(\phi_1 - \theta\right)\right] \\ & - \frac{\mu_o J r}{4\pi} ln\left(\frac{\rho_2}{\rho_1}\right)\left[sin\left(2\left(\phi_2 - \theta\right)\right) - sin\left(2\left(\phi_1 - \theta\right)\right)\right] \\ & + \frac{\mu_o J}{2\pi}\sum_{n=3}^{\infty}\frac{r^{n-1}}{n\left(n-2\right)}\left(\frac{1}{\rho_2^{n-2}} - \frac{1}{\rho_1^{n-2}}\right) \times \\ & \left[sin\left(n\left(\phi_2 - \theta\right)\right) - sin\left(n\left(\phi_1 - \theta\right)\right)\right]. \end{aligned} \tag{1.5.78b}$$

Now the harmonics components a_n and b_n (the dimensionless coefficients as defined in Eqs. (1.5.31)) are computed due to the field from a single current block. It should be noted that the summation of a_n and b_n starts from $n = 0$ instead of $n = 1$ in Eq. (1.5.78). For $n > 1$ and harmonics normalized to the dipole field, the following expressions for the normal and skew harmonics at a reference radius R_o are obtained using the procedure of Eqs. (1.5.42) :

$$b_n = \frac{-10^4 R_0^n}{(n^2-1)}\left[\left(\frac{1}{\rho_2^{n-1}} - \frac{1}{\rho_1^{n-1}}\right)/(\rho_2 - \rho_1)\right] \frac{sin\left((n+1)\,\phi_2\right) - sin\left((n+1)\,\phi_1\right)}{sin\left(\phi_2\right) - sin\left(\phi_1\right)}, \tag{1.5.79a}$$

$$a_n = \frac{-10^4 R_0^n}{(n^2-1)}\left[\left(\frac{1}{\rho_2^{n-1}} - \frac{1}{\rho_1^{n-1}}\right)/(\rho_2 - \rho_1)\right] \frac{cos\left((n+1)\,\phi_2\right) - cos\left((n+1)\,\phi_1\right)}{sin\left(\phi_2\right) - sin\left(\phi_1\right)}, \tag{1.5.79b}$$

and the harmonic expressions for $n = 1$ are

$$b_n = \frac{10^4 R_0 ln\left(\frac{\rho_2}{\rho_1}\right)}{\rho_2 - \rho_1} \frac{sin\left(2\phi_2\right) - sin\left(2\phi_1\right)}{sin\left(\phi_2\right) - sin\left(\phi_1\right)}, \tag{1.5.80a}$$

$$a_n = \frac{10^4 R_0 ln\left(\frac{\rho_2}{\rho_1}\right)}{\rho_2 - \rho_1} \frac{cos\left(2\phi_2\right) - cos\left(2\phi_1\right)}{sin\left(\phi_2\right) - sin\left(\phi_1\right)}. \tag{1.5.80b}$$

To compute A_n and B_n (having the dimensions of field and defined in Eqs. (1.5.21)) one derives the expressions for field components from Eqs. (1.5.78) at a reference radius R_o in the form of :

$$B_r = \sum_{n=1}^{\infty}\left(\frac{r}{R_o}\right)^{n-1}\left[B_n \;\; sin\left(n\theta\right) \;\; + \;\; A_n \;\; cos\left(n\theta\right)\right], \tag{1.5.81a}$$

$$B_\theta = \sum_{n=1}^{\infty}\left(\frac{r}{R_o}\right)^{n-1}\left[B_n \;\; cos\left(n\theta\right) \;\; - \;\; A_n \;\; sin\left(n\theta\right)\right], \tag{1.5.81b}$$

to obtain

$$A_1 = -\frac{\mu_o J}{2\pi}(\rho_2 - \rho_1)\left[cos\left(\phi_2\right) - cos\left(\phi_1\right)\right], \tag{1.5.82a}$$

$$A_2 = -\frac{\mu_o J R_o}{2\pi} ln\left(\frac{\rho_2}{\rho_1}\right)\left[cos\left(2\phi_2\right) - cos\left(2\phi_1\right)\right], \tag{1.5.82b}$$

for n $\geq$ 3

$$A_n = \frac{\mu_o J}{2\pi}\frac{R_o^{n-1}}{n\,(n-2)}\left(\frac{1}{\rho_2^{n-2}} - \frac{1}{\rho_1^{n-2}}\right)\left[cos\left(n\phi_2\right) - cos\left(n\phi_1\right)\right], \tag{1.5.82c}$$

and

$$B_1 = -\frac{\mu_o J}{2\pi}(\rho_2 - \rho_1)\left[sin\left(\phi_2\right) - sin\left(\phi_1\right)\right], \tag{1.5.83a}$$

$$B_2 = -\frac{\mu_o J R_o}{2\pi} ln\left(\frac{\rho_2}{\rho_1}\right)\left[sin\left(2\phi_2\right) - sin\left(2\phi_1\right)\right], \tag{1.5.83b}$$

for n $\geq$ 3

$$B_n = \frac{\mu_o J}{2\pi}\sum_{n=3}^{\infty}\frac{R_o^{n-1}}{n\,(n-2)}\left(\frac{1}{\rho_2^{n-2}} - \frac{1}{\rho_1^{n-2}}\right)\left[sin\left(n\phi_2\right) - sin\left(n\phi_1\right)\right]. \tag{1.5.83c}$$

In a typical superconducting magnet several current blocks are used to generate the desired multipolar field. In order to compute the harmonics due to several current blocks, the field and field harmonics A_n and B_n (coefficients having the dimension of field) can be directly superimposed. However, a_n and b_n (dimensionless coefficients) can not be directly added and they must be obtained from A_n and B_n as follows :

$$b_n = 10^4 \frac{\sum_k (B_{n+1})_k}{\sum_k (B_{m+1})_k}, \tag{1.5.84a}$$

$$a_n = 10^4 \frac{\sum_k (A_{n+1})_k}{\sum_k (B_{m+1})_k}, \tag{1.5.84b}$$

where the summation k is carried over all k blocks with the k^{th} block carrying a current density of J_k and located between radii ρ_{1k} and ρ_{2k} and angles ϕ_{1k} and ϕ_{2k}. The A_n and B_n for each current blocks are computed using the expressions given above. The harmonics are defined such that the fundamental harmonic b_m is normalized to 10^4.

The field components outside a current block ($r > \rho_2$) are obtained similarly by integrating Eqs. (1.5.78) and the results are given below

$$B_r(r,\theta) = -\frac{\mu_o J}{2\pi} \sum_{n=1}^{\infty} \frac{\rho_2^{n+1} - \rho_1^{n+1}}{n(n+2)\,r^{n+1}} \left[cos\left(n\left(\phi_2 - \theta\right)\right) - cos\left(n\left(\phi_1 - \theta\right)\right)\right], \tag{1.5.85a}$$

$$B_\theta(r,\theta) = \frac{\mu_o J}{2\pi} \sum_{n=1}^{\infty} \frac{\rho_2^{n+1} - \rho_1^{n+1}}{n(n+2)\,r^{n+1}} \left[sin\left(n\left(\phi_2 - \theta\right)\right) - sin\left(n\left(\phi_1 - \theta\right)\right)\right]. \tag{1.5.85b}$$

The field inside a current block ($\rho_1 < r < \rho_2$) can be obtained by dividing the current block in two parts (a) from radius ρ_1 to radius r and (b) from radius r to radius ρ_2. Then the superimposition principle can be used to determine the field components with the (a) part evaluated from Eqs. (1.5.78) with ρ_2 replaced by r and the (b) part from Eqs. (1.5.85) with ρ_1 replaced by r.

1.5.3.5. Field Harmonics due to Current Blocks in a Cylindrical Iron Shell

As shown in a previous section (Eqs. (1.5.67) for $r < a$), the expressions for the field component due to current blocks get modified when they are placed inside an iron shell having an iron inner radius of R_f and outer radius of R_a. The harmonic coefficients A_n and B_n are enhanced by :

$$K_n = \left[1 + \frac{\mu - 1}{\mu + 1}\left(\frac{a}{R_f}\right)^{2n}\right] \frac{\left[1 - \left(\frac{R_f}{R_a}\right)^{2n}\right]}{\left[1 - \left(\frac{\mu-1}{\mu+1}\right)^2 \left(\frac{R_f}{R_a}\right)^{2n}\right]},$$

To give

$$A_n^{'} = K_n \times A_n,$$

and

$$B_n^{'} = K_n \times B_n.$$

The harmonics coefficients a_n and b_n given in Eqs. (1.5.84) are modified to :

$$b_n = 10^4 \frac{\sum_k \left(K_{n+1} \;\; B_{n+1}\right)_k}{\sum_k \left(K_{m+1} \;\; B_{m+1}\right)_k} \qquad (1.5.86a)$$

$$a_n = 10^4 \frac{\sum_k \left(K_{n+1} \;\; A_{n+1}\right)_k}{\sum_k \left(K_{m+1} \;\; B_{m+1}\right)_k} \qquad (1.5.86b)$$

1.5.3.6. COS($m\theta$) Current Distribution for Ideal Fields

In this section, it is demonstrated that an ideal $2m$ (m=1 for dipole) multipolar field shape in accelerator magnets can be produced by a COS($m\theta$) current distribution. In the last section the expressions for the field and vector potential produced by a line current were obtained. The field in the cross section of the magnet can be described by superimposing the field produced by a large number of such wires.

A cylindrical current sheet [12-18] at a radius of a is shown in Fig. 1.5.2, where the angular current density $I(\phi)$ in *Amperes/radian* as a function of angle ϕ is given by the relation

$$I(\phi) \quad = \quad I_o \; cos(m\phi). \tag{1.5.87}$$

[In the case of skew harmonics the current distribution is $I(\phi) \; = \; I_o \; sin(m\phi)$].

It will be demonstrated that a pure dipole field is created by m=*1*, quadrupole by m=*2*, sextupole by m=*3*, etc. The total current required (Ampere-turns) per pole for generating a $2m$-pole field is given by

$$I_{pole} \quad = \quad \int_o^{\pi/2m} I_o \; cos(m\phi) \; d\phi \quad = \quad \frac{I_o}{m}.$$

In Eqs. (1.5.39), the vector potential produced by a single wire at any position is computed. To obtain the vector potential at (r,θ) inside the sheet (i.e. r < a), the expression is integrated over ϕ

$$A_z(r,\theta) = \frac{\mu_o I_o}{2\pi} \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)\left(\frac{r}{a}\right)^n \int_o^{2\pi} cos(m\phi) \;\; cos(n(\phi-\theta)) \;\; d\phi, \tag{1.5.88}$$

to obtain

$$A_z(r,\theta) = \frac{\mu_o I_o}{2m}\left(\frac{r}{a}\right)^m \;\; cos(m\theta), \tag{1.5.89}$$

where the following trigonometric relations have been used

$$cos[n(\phi-\theta)] = cos(n\phi) \;\; cos(n\theta) + sin(n\phi) \;\; sin(n\theta), \tag{1.5.90}$$

$$\int_o^{2\pi} cos(m\phi) \;\; cos(n\phi) \;\; d\phi = \pi\delta_{m,n}, \tag{1.5.91a}$$

$$\int_o^{2\pi} cos(m\phi) \;\; sin(n\phi) \;\; d\phi = 0. \tag{1.5.91b}$$

The field components inside the current sheet are obtained by using Eqs. (1.5.40)

$$B_\theta(r,\theta) = -\frac{\mu_o I_o}{2a}\left(\frac{r}{a}\right)^{m-1} \;\; cos(m\theta), \tag{1.5.92a}$$

$$B_r(r,\theta) = -\frac{\mu_o I_o}{2a}\left(\frac{r}{a}\right)^{m-1} \;\; sin(m\theta), \tag{1.5.92b}$$

$$B_z(r,\theta) = 0. \tag{1.5.92c}$$

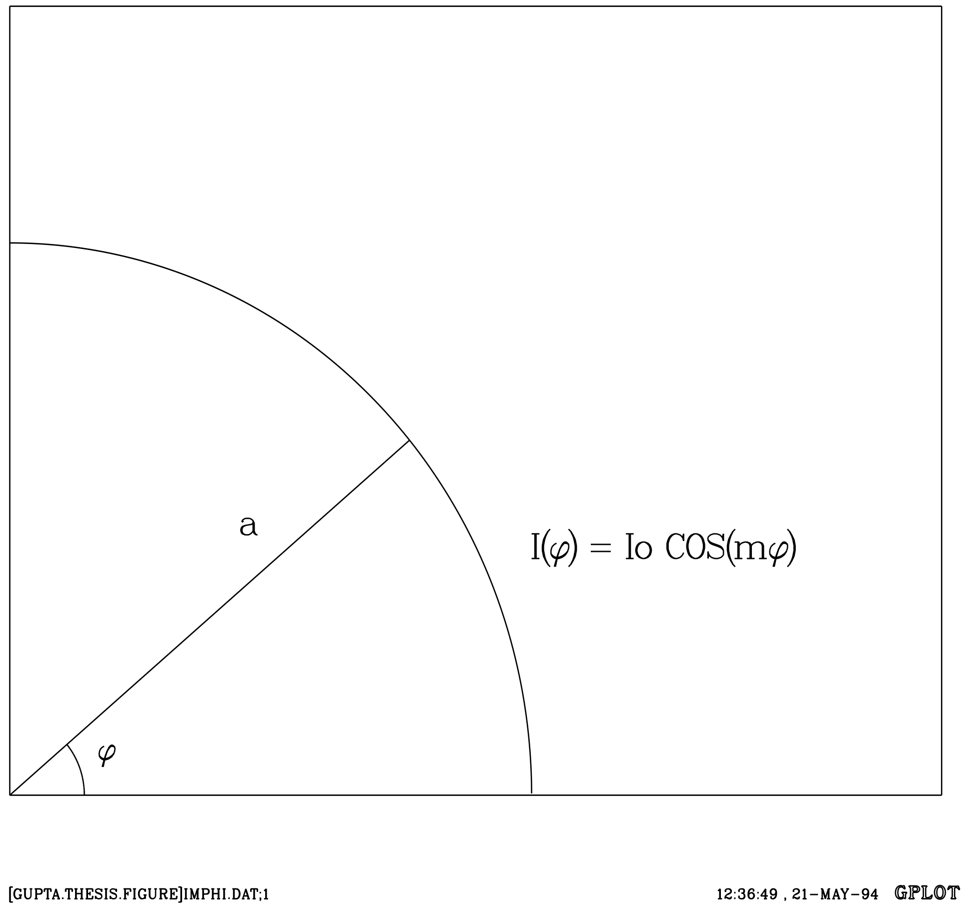

Figure 1.5.2: Computation of the field at (r, θ) produced by a current sheet at a radius a in which the current density varies as a function of angle given by $I(\phi) \;=\; I_o \; cos(m\phi)$.

It may be noted that the magnitude of the field $|B|$ is independent of θ. On using Eqs. (1.5.46)

$$B_x\,(r,\theta) \quad = \quad -\,\frac{\mu_o I_o}{2a}\,\left(\frac{r}{a}\right)^{m-1} sin\,((m-1)\,\theta)\,, \tag{1.5.93a}$$

$$B_y\,(r,\theta) \quad = \quad -\,\frac{\mu_o I_o}{2a}\left(\frac{r}{a}\right)^{m-1} cos\,((m-1)\,\theta)\,. \tag{1.5.93b}$$

For the m=1 case, this generates a pure dipole field, as the field components from Eqs. (1.5.92) reduce to

$$B_\theta(r,\theta) = -\frac{\mu_o I_o}{2a} cos(\theta),$$
$$B_r(r,\theta) = -\frac{\mu_o I_o}{2a} sin(\theta),$$

and, from Eqs. (1.5.93)

$$B_x = 0, \quad (1.5.94a)$$
$$B_y = -\frac{\mu_o I_o}{2a}. \quad (1.5.94b)$$

This implies that a cylindrical current sheet with a cosine θ current distribution would create a uniform vertical field inside it. This basic result is widely used in designing superconducting accelerator dipole magnets, although the actual current distribution is somewhat modified for practical reasons.

Likewise, for m=2, a pure quadrupole field is generated

$$B_\theta(r,\theta) = -\frac{\mu_o I_o r}{2a^2} cos(2\theta),$$
$$B_r(r,\theta) = -\frac{\mu_o I_o r}{2a^2} sin(2\theta),$$

and, from Eqs. (1.5.93)

$$B_x = g\ y, \quad (1.5.95a)$$
$$B_y = g\ x, \quad (1.5.95b)$$

with $g = -(\mu_o I_o)/(2a^2)$.

Similarly, for m=3, a pure sextupole field is generated

$$B_\theta(r,\theta) = -\frac{\mu_o I_o r^2}{2a^3} cos(3\theta),$$
$$B_r(r,\theta) = -\frac{\mu_o I_o r^2}{2a^3} sin(3\theta),$$

and, from Eqs. (1.5.93)

$$B_x = 2S\ x\ y, \quad (1.5.96a)$$
$$B_y = S\left(x^2 - y^2\right), \quad (1.5.96b)$$

with $S = -(\mu_o I_o)/(2a^3)$.

In general, a $cos(m\theta)$ current distribution gives a $2m$ order multipole with field components given by Eqs. (1.5.93).

On the x-axis (midplane), $\theta = 0$, these components become

$$B_x\left(x,0\right) = 0, \tag{1.5.97a}$$

$$B_y\left(x,0\right) = -\frac{\mu_o I_o}{2a}\left(\frac{x}{a}\right)^{m-1}, \tag{1.5.97b}$$

and on the y-axis

$$\begin{aligned} B_x\left(0,y\right) &= 0, && for \quad m = 1,3,5,... \\ &= \pm\frac{\mu_o I_o}{2a}\left(\frac{y}{a}\right)^{m-1}, && for \quad m = 2,4,6,... \end{aligned} \tag{1.5.98a}$$

$$\begin{aligned} B_y\left(0,y\right) &= \pm\frac{\mu_o I_o}{2a}\left(\frac{y}{a}\right)^{m-1}, && for \quad m = 1,3,5,... \\ &= 0. && for \quad m = 2,4,6,... \end{aligned} \tag{1.5.98b}$$

To obtain the field outside the current sheet (r $>$ a), Eqs. (1.5.44) is integrated using the trigonometric relations given in Eq. (1.5.90)and Eqs. (1.5.91)

$$\begin{aligned} A_z\left(r,\theta\right) = &-\frac{\mu_o I_o}{2\pi} ln\left(\frac{r}{a}\right)\int_o^{2\pi} cos\left(m\phi\right)\ d\phi \\ &+ \frac{\mu_o I_o}{2\pi}\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)\left(\frac{a}{r}\right)^n \int_o^{2\pi} cos\left(m\phi\right)\ cos\left(n\left(\phi-\theta\right)\right)\ d\phi, \end{aligned}$$

$$\text{therefore,}\quad A_z\left(r,\theta\right) = \frac{\mu_o I_o}{2m}\left(\frac{a}{r}\right)^m cos\left(m\theta\right). \tag{1.5.99}$$

The field components for r $>$ a are obtained using Eqs. (1.5.40)

$$B_\theta\left(r,\theta\right) = \frac{\mu_o I_o}{2a}\left(\frac{a}{r}\right)^{m+1} cos\left(m\theta\right), \tag{1.5.100a}$$

$$B_r\left(r,\theta\right) = -\frac{\mu_o I_o}{2a}\left(\frac{a}{r}\right)^{m+1} sin\left(m\theta\right), \tag{1.5.100b}$$

$$B_z\left(r,\theta\right) = 0, \tag{1.5.100c}$$

and the (B_x, B_y) components of the field are obtained as :

$$B_x = B_r\ cos\left(\theta\right) - B_\theta\ sin\left(\theta\right),$$

and

$$B_y = B_r\ sin\left(\theta\right) + B_\theta\ cos\left(\theta\right),$$

therefore,

$$B_x = -\frac{\mu_o I_o}{2a}\left(\frac{a}{r}\right)^{m+1} sin\left[\left(m+1\right)\theta\right], \tag{1.5.101a}$$

$$B_y = \frac{\mu_o I_o}{2a}\left(\frac{a}{r}\right)^{m+1} cos\left[\left(m+1\right)\theta\right]. \tag{1.5.101b}$$

In the case of the dipole (m=1), the field components outside the current sheet, fall as $\frac{1}{r^2}$, and are given by :

$$B_\theta\left(r,\theta\right) = \frac{\mu_o I_o a}{2r^2} cos\left[\theta\right], \tag{1.5.102a}$$

$$B_r\left(r,\theta\right) = -\frac{\mu_o I_o a}{2r^2} sin\left[\theta\right], \tag{1.5.102b}$$

$$B_x\left(r,\theta\right) = -\frac{\mu_o I_o a}{2r^2} sin\left[2\theta\right], \tag{1.5.102c}$$

$$B_y\left(r,\theta\right) = \frac{\mu_o I_o a}{2r^2} cos\left[2\theta\right]. \tag{1.5.102d}$$

In deriving the above expressions, for simplicity it is assumed that the current is localized in a sheet. However, in accelerator magnets, the current is present between two radii a_1 and a_2. It is assumed that the current density in Amperes/m^2 is given by

$$J\left(\phi\right) \quad = \quad J_o \ cos\left(m\phi\right).$$

For a sheet of infitesimal thickness da, J_o is related to the angular current density (I_o) as

$$I_o \quad = \quad J_o \ a \ da,$$

In this case the expression for the vector potential and field components for r < a are by integrating Eqs. (1.5.39):

$$A_z\left(r,\theta\right) = \frac{\mu_o J_o}{2\pi}\sum_{n=1}^{\infty}\left(\frac{r^n}{n}\right)\int_{a_1}^{a_2}\frac{1}{a^n}a \ \ da\int_o^{2\pi} \ \ cos\left(m\phi\right) \ \ cos\left(n\left(\phi-\theta\right)\right) \ \ d\phi,$$

Therefore,

$$A_z\left(r,\theta\right) = \frac{\mu_o J_o r^m}{2m} \ \ cos\left(m\theta\right)\int_{a_1}^{a_2}\frac{1}{a^{m-1}} \ \ da, \tag{1.5.103}$$

$$B_\theta\left(r,\theta\right) = -\frac{\mu_o J_o r^{m-1}}{2} \ \ cos\left(m\theta\right)\int_{a_1}^{a_2}\frac{1}{a^{m-1}} \ \ da, \tag{1.5.104a}$$

$$B_r\left(r,\theta\right) = -\frac{\mu_o J_o r^{m-1}}{2} \ \ sin\left(m\theta\right)\int_{a_1}^{a_2}\frac{1}{a^{m-1}} \ \ da, \tag{1.5.104b}$$

$$B_z\left(r,\theta\right) = 0. \tag{1.5.104c}$$

Except for $m = 2$ case (the quadrupole case, for which the expressions are given later), one obtains :

$$A_z\left(r,\theta\right) = \frac{\mu_o J_o {a_1}^2}{2m\left(m-2\right)} \ \ cos\left(m\theta\right)\left(\frac{r}{a_1}\right)^m\left(1-\left(\frac{a_1}{a_2}\right)^{m-2}\right), \tag{1.5.105}$$

$$B_\theta\left(r,\theta\right) = -\frac{\mu_o J_o a_1}{2\left(m-2\right)} \ \ cos\left(m\theta\right)\left(\frac{r}{a_1}\right)^{m-1}\left(1-\left(\frac{a_1}{a_2}\right)^{m-2}\right), \tag{1.5.106a}$$

$$B_r(r,\theta) = -\frac{\mu_o J_o a_1}{2(m-2)}\ sin(m\theta)\left(\frac{r}{a_1}\right)^{m-1}\left(1-\left(\frac{a_1}{a_2}\right)^{m-2}\right), \tag{1.5.106b}$$

$$B_y(r,\theta) = -\frac{\mu_o J_o a_1}{2(m-2)}\ cos((m-1)\theta)\left(\frac{r}{a_1}\right)^{m-1}\left(1-\left(\frac{a_1}{a_2}\right)^{m-2}\right), \tag{1.5.106c}$$

$$B_x(r,\theta) = -\frac{\mu_o J_o a_1}{2(m-2)}\ sin((m-1)\theta)\left(\frac{r}{a_1}\right)^{m-1}\left(1-\left(\frac{a_1}{a_2}\right)^{m-2}\right). \tag{1.5.106d}$$

In the case of the dipole (m=1), this gives a vertical field

$$B_y = -\mu_o J_o\left(\frac{a_2-a_1}{2}\right) = -\mu_o J_o\left(\frac{\Delta a}{2}\right).$$

For $m = 2$ (quadrupole), the integration of Eqs. (1.5.104) gives :

$$A_z(r,\theta) = \frac{\mu_o J_o r^2}{4}\ cos(2\theta)\ \ ln\left(\frac{a_2}{a_1}\right) \tag{1.5.107}$$

$$B_\theta(r,\theta) = -\frac{\mu_o J_o r}{2}\ cos(2\theta)\ \ ln\left(\frac{a_2}{a_1}\right) \tag{1.5.108a}$$

$$B_r(r,\theta) = -\frac{\mu_o J_o r}{2}\ sin(2\theta)\ \ ln\left(\frac{a_2}{a_1}\right) \tag{1.5.108b}$$

$$B_y(r,\theta) = -\frac{\mu_o J_o r}{2}\ cos(\theta)\ \ ln\left(\frac{a_2}{a_1}\right) \tag{1.5.108c}$$

$$B_x(r,\theta) = -\frac{\mu_o J_o r}{2}\ sin(\theta)\ \ ln\left(\frac{a_2}{a_1}\right) \tag{1.5.108d}$$

If the sheet thickness $\Delta a = a_2 - a_1$ is very small compared to the the average radius $\bar{a} = \frac{(a_2+a_1)}{2}$, then the expressions in Eqs. (1.5.106) for $r < a$ may be simplified to the following equations since the integral in Eq. (1.5.103) and Eqs. (1.5.104) can be approximated as $(\Delta a/\bar{a}^{m-1})$:

$$A_z(r,\theta) = \frac{\mu_o J_o r\Delta a}{2m}\left(\frac{r}{\bar{a}}\right)^{m-1}\ cos(m\theta), \tag{1.5.109}$$

$$B_\theta(r,\theta) = -\frac{\mu_o J_o \Delta a}{2}\left(\frac{r}{\bar{a}}\right)^{m-1}\ cos(m\theta), \tag{1.5.110a}$$

$$B_r(r,\theta) = -\frac{\mu_o J_o \Delta a}{2}\left(\frac{r}{\bar{a}}\right)^{m-1}\ sin(m\theta). \tag{1.5.110b}$$

1.5.3.7. COS($m\theta$) Current Distribution in a Cylindrical Iron Shell

In superconducting accelerator magnets, the coils are frequently placed inside a cylindrical iron yoke to (a) reduce the stray magnetic field outside the magnet and (b) as an added benefit to enhance the field in the aperture of the magnet. Due to the non-linear properties of the iron, the fraction of field generated by the iron at any current depends on how much the yoke is magnetized. This is too complex a problem to solve analytically. However, one can obtain simple expressions if one assumes that the permeability (μ) of the iron is constant everywhere in the yoke. Expressions for the vector potential and the field are given for the case in which a $COS(m\theta)$ current sheet at radius a is inside in an iron shell with inner radius of R_f and outer radius of R_a.

In this case, the method of matching the boundary conditions at the air and iron interfaces, as described in the last section, can be used to include the contribution from the iron. This is equivalent to the method of images when the effect of the iron is replaced by the equivalent image currents.

In the presence of a cylindrical iron yoke, the vector potential and the field components given in Eqs. (1.5.89) and Eqs. (1.5.92), for $r < a$, are modified to

$$A_z\left(r,\theta\right) = \frac{\mu_o I_o}{2m}\ cos\left(m\theta\right)\left(\frac{r}{a}\right)^m \quad \times$$
$$\left[1+\frac{\mu-1}{\mu+1}\left(\frac{a}{R_f}\right)^{2m}\frac{\left[1-\left(\frac{R_f}{R_a}\right)^{2m}\right]}{\left[1-\left(\frac{\mu-1}{\mu+1}\right)^2\left(\frac{R_f}{R_a}\right)^{2m}\right]}\right], \tag{1.5.111}$$

$$B_\theta\left(r,\theta\right) = -\frac{\mu_o I_o}{2a}\ cos\left(m\theta\right)\left(\frac{r}{a}\right)^{m-1} \quad \times$$
$$\left[1+\frac{\mu-1}{\mu+1}\left(\frac{a}{R_f}\right)^{2m}\frac{\left[1-\left(\frac{R_f}{R_a}\right)^{2m}\right]}{\left[1-\left(\frac{\mu-1}{\mu+1}\right)^2\left(\frac{R_f}{R_a}\right)^{2m}\right]}\right], \tag{1.5.112a}$$

$$B_r\left(r,\theta\right) = -\frac{\mu_o I_o}{2a}\ sin\left(m\theta\right)\left(\frac{r}{a}\right)^{m-1} \quad \times$$
$$\left[1+\frac{\mu-1}{\mu+1}\left(\frac{a}{R_f}\right)^{2m}\frac{\left[1-\left(\frac{R_f}{R_a}\right)^{2m}\right]}{\left[1-\left(\frac{\mu-1}{\mu+1}\right)^2\left(\frac{R_f}{R_a}\right)^{2m}\right]}\right]. \tag{1.5.112b}$$

The other components are obtained using Eqs. (1.5.46)

$$B_x\left(r,\theta\right) = -\frac{\mu_o I_o}{2a}\ sin\left(\left(m-1\right)\theta\right)\left(\frac{r}{a}\right)^{m-1} \quad \times$$

$$\left[1+\frac{\mu-1}{\mu+1}\left(\frac{a}{R_f}\right)^{2m}\frac{\left[1-\left(\frac{R_f}{R_a}\right)^{2m}\right]}{\left[1-\left(\frac{\mu-1}{\mu+1}\right)^2\left(\frac{R_f}{R_a}\right)^{2m}\right]}\right], \tag{1.5.113a}$$

$$B_y\left(r,\theta\right)=-\frac{\mu_o I_o}{2a}\ cos\left(\left(m-1\right)\theta\right)\left(\frac{r}{a}\right)^{m-1}\quad\times$$

$$\left[1+\frac{\mu-1}{\mu+1}\left(\frac{a}{R_f}\right)^{2m}\frac{\left[1-\left(\frac{R_f}{R_a}\right)^{2m}\right]}{\left[1-\left(\frac{\mu-1}{\mu+1}\right)^2\left(\frac{R_f}{R_a}\right)^{2m}\right]}\right], \tag{1.5.113b}$$

$$B_z\left(r,\theta\right)=0. \tag{1.5.113c}$$

Similarly, the vector potential and field outside the current sheet but inside the iron, i.e. $a < r < R_f$, is given by :

Between Coil and Iron ($\mathbf{a < r < R_f}$)

$$A_z\left(r,\theta\right)=\frac{\mu_o I_o}{2m}\left(1+\frac{\mu-1}{\mu+1}\left(\frac{r}{R_f}\right)^{2m}\frac{\left[1-\left(\frac{R_f}{R_a}\right)^{2m}\right]}{\left[1-\left(\frac{\mu-1}{\mu+1}\right)^2\left(\frac{R_f}{R_a}\right)^{2m}\right]}\right)\quad\times$$

$$\left(\frac{a}{r}\right)^{m} cos\left(m\theta\right), \tag{1.5.114}$$

$$B_r=-\frac{\mu_o I_o}{2a}\left(1+\frac{\mu-1}{\mu+1}\left(\frac{r}{R_f}\right)^{2m}\frac{\left[1-\left(\frac{R_f}{R_a}\right)^{2m}\right]}{\left[1-\left(\frac{\mu-1}{\mu+1}\right)^2\left(\frac{R_f}{R_a}\right)^{2m}\right]}\right)\quad\times$$

$$\left(\frac{a}{r}\right)^{m+1} sin\left(m\theta\right), \tag{1.5.115a}$$

$$B_\theta=\frac{\mu_o I_o}{2a}\left(1-\frac{\mu-1}{\mu+1}\left(\frac{r}{R_f}\right)^{2m}\frac{\left[1-\left(\frac{R_f}{R_a}\right)^{2m}\right]}{\left[1-\left(\frac{\mu-1}{\mu+1}\right)^2\left(\frac{R_f}{R_a}\right)^{2m}\right]}\right)\quad\times$$

$$\left(\frac{a}{r}\right)^{m+1} cos\left(m\theta\right). \tag{1.5.115b}$$

Inside Iron ($\mathbf{R_f < r < R_a}$)

$$A_z\left(r,\theta\right)=\frac{\mu\mu_o I_o}{m\left(\mu+1\right)}\left(\frac{1-\frac{\mu-1}{\mu+1}\left(\frac{r}{R_a}\right)^{2m}}{1-\left(\frac{\mu-1}{\mu+1}\right)^2\left(\frac{R_f}{R_a}\right)^{2m}}\right)\quad\times$$

$$\left(\frac{a}{r}\right)^{m} cos\left(m\theta\right), \tag{1.5.116}$$

$$B_r=-\frac{\mu\mu_o I_o}{a\left(\mu+1\right)}\left(\frac{1-\frac{\mu-1}{\mu+1}\left(\frac{r}{R_a}\right)^{2m}}{1-\left(\frac{\mu-1}{\mu+1}\right)^2\left(\frac{R_f}{R_a}\right)^{2m}}\right)\quad\times$$

$$\left(\frac{a}{r}\right)^{m+1} sin\left(m\theta\right), \tag{1.5.117a}$$

$$B_\theta = \frac{\mu\mu_o I_o}{a\left(\mu+1\right)} \left(\frac{1+\frac{\mu-1}{\mu+1}\left(\frac{r}{R_a}\right)^{2m}}{1-\left(\frac{\mu-1}{\mu+1}\right)^2\left(\frac{R_f}{R_a}\right)^{2m}} \right) \times$$

$$\left(\frac{a}{r}\right)^{m+1} cos\left(m\theta\right). \tag{1.5.117b}$$

<u>Outside Iron ($\mathbf{r > R_a}$)</u>

$$A_z\left(r,\theta\right) = \frac{2\mu\mu_o I_o}{m\left(\mu+1\right)^2} \frac{1}{1-\left(\frac{\mu-1}{\mu+1}\right)^2\left(\frac{R_f}{R_a}\right)^{2m}} \left(\frac{a}{r}\right)^m cos\left(m\theta\right), \tag{1.5.118}$$

$$B_r = -\frac{2\mu\mu_o I_o}{a\left(\mu+1\right)^2} \frac{1}{1-\left(\frac{\mu-1}{\mu+1}\right)^2\left(\frac{R_f}{R_a}\right)^{2m}} \left(\frac{a}{r}\right)^{m+1} sin\left(m\theta\right), \tag{1.5.119a}$$

$$B_\theta = \frac{2\mu\mu_o I_o}{a\left(\mu+1\right)^2} \frac{1}{1-\left(\frac{\mu-1}{\mu+1}\right)^2\left(\frac{R_f}{R_a}\right)^{2m}} \left(\frac{a}{r}\right)^{m+1} cos\left(m\theta\right). \tag{1.5.119b}$$

1.5.3.8. Intersecting Circles with a Constant Current Density for Ideal Fields

It has been shown [137] that a pure dipole field can be created simply by two intersecting circles carrying constant current densities in opposite directions. To demonstrate this, the field is evaluated inside and outside a circular conductor with a radius a and carrying a constant current density J in the direction of the axis (perpendicular to the plane of paper). For a radius $R > a$ (outside the conductor), Ampere's law gives

$$2\pi R \cdot H = \pi a^2 \ J.$$

Therefore,

$$H = \frac{Ja^2}{2R}. \tag{1.5.120}$$

The direction of the magnetic field is azimuthal, with (x, y) components of the field at any point outside the conductor given by

$$\begin{aligned} H_x &= -\frac{Ja^2}{2R} sin\,(\theta) = -\frac{J}{2}\left(\frac{a}{R}\right)^2 y, \\ H_y &= \frac{Ja^2}{2R} cos\,(\theta) \quad = \frac{J}{2}\left(\frac{a}{R}\right)^2 x. \end{aligned}$$

The field inside the conductor $(R < a)$ can be obtained as

$$2\pi R \cdot H = \pi R^2 \ J,$$

$$i.e., \qquad H = \frac{JR}{2}. \tag{1.5.121}$$

with the components of the field being given by

$$\begin{aligned} H_x &= -\frac{J}{2} R \ sin\,(\theta) = -\frac{J}{2} y, \\ H_y &= \frac{J}{2} R \ cos\,(\theta) \quad = \frac{J}{2} x. \end{aligned}$$

Now expressions will be derived for the field produced by the conductors in two intersecting circles. The coordinate system is defined such that the x-axis passes through the centers of the two circles with the origin of the new coordinate system (x', y') being in the middle of the two. The distance between the centers of the two circles is s with circle 2 to the right such that $x' = x_1 - \frac{s}{2} = x_2 + \frac{s}{2}$ and $y_1 = y_2 = y'$. The direction of the current is opposite in the two circles, with constant current densities J_1 and $-J_2$ respectively. The components of the field inside the region created by the two intersecting circles can be computed by superimposing the field produced by the conductors in the two circles

$$\begin{aligned} H_x &= \frac{y'}{2}\left(J_2 - J_1\right), \\ H_y &= \frac{x'}{2}\left(J_1 - J_2\right) + \frac{s}{4}\left(J_1 + J_2\right). \end{aligned}$$

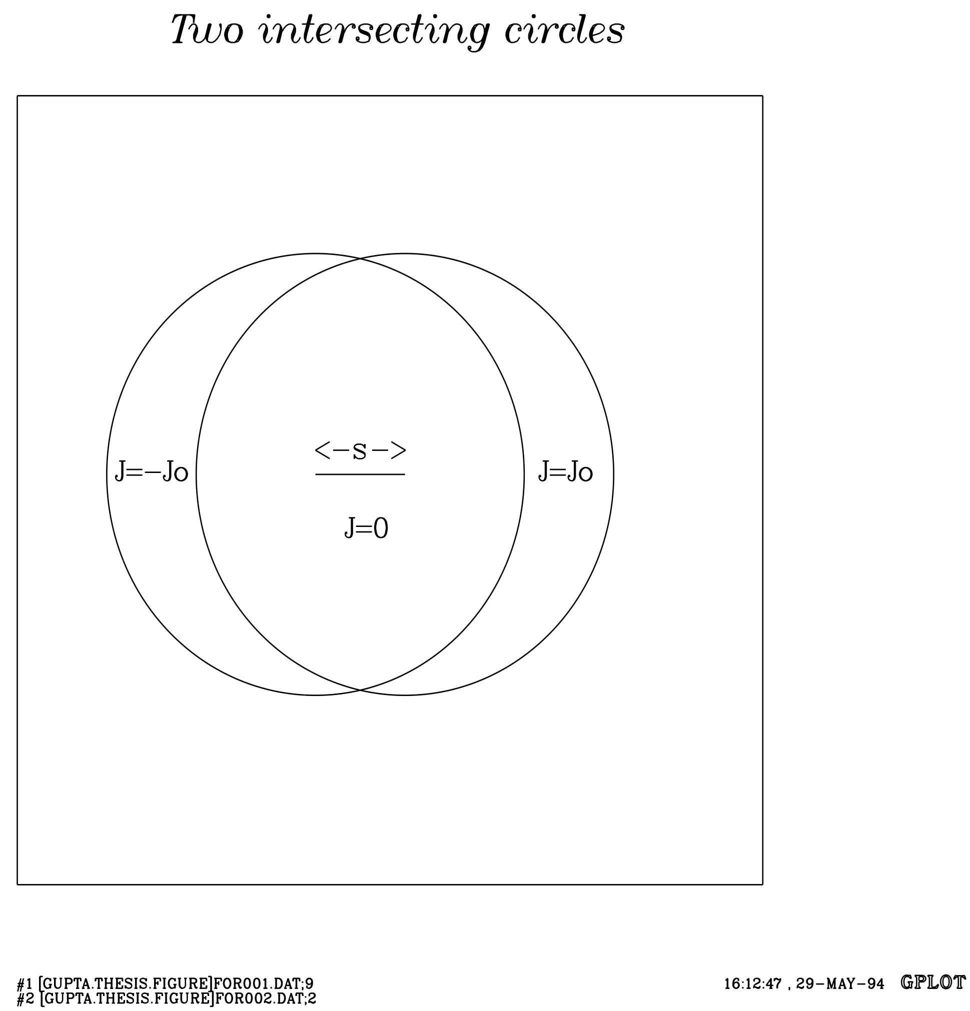

Figure 1.5.3: This figure shows the two intersecting circles of equal size with one carrying a current with a constant density $J = -J_o$ and the other $J = J_o$. The two circles are separated by a distance s. In the intersection region of the two circles, the net current density is zero and therefore it can be replaced by a current free region. It is demonstrated that this configuration produces a vertical dipole field given by $\frac{J_o}{2}s$.

A special case comes when the magnitude of the current densities in the two circles is J_o but the direction is opposite as shown in Fig. 1.5.3. This means that the intersection region is a current free region which can be used as an aperture for the particle beam and and the aperture has a constant vertical magnetic field given by $H_y = \frac{J_o}{2}s$.

It can be shown [14] that four intersecting circles create a quadrupole field and in general $2m$ intersecting circles create a $2m$-order multipole. The treatment has been also been extended to ellipses by a number of authors (see for example Beth [14]).

1.5.4. Complex Variable Method in 2-d Magnetic Field Calculations

The method of complex variable is found very useful in deriving many expressions in superconducting magnets. [12-18,81] These methods can be applied to 2-dimensional field computations, which is the case for the most part of long superconducting magnets. Mills and Morgan [115] have shown that the complex method can also be extended throughout the ends, however, to the field integral ($\int B.dz$). The complex variables have two parts (real and imaginary) and the following variables will be used :

$$z = x + i\ y, \tag{1.5.122a}$$

$$H(z) = H_y + i\ H_x, \tag{1.5.122b}$$

$$B(z) = B_y + i\ B_x, \tag{1.5.122c}$$

$$W(z) = -(A + i\ \phi) + constant. \tag{1.5.122d}$$

where W is the complex potential having ϕ and A (scalar and vector potentials) as the two components, and $i = \sqrt{-1}$. z^* is the complex conjugate of z with

$$z^* = x - i\ y.$$

In the 2-d case the following relations are valid :

$$B_x = \frac{\partial A}{\partial y} \tag{1.5.123a}$$

$$B_y = -\frac{\partial A}{\partial x}, \tag{1.5.123b}$$

$$with \quad B_x = \mu_0 \mu H_x,$$

$$and \quad B_y = \mu_0 \mu H_y.$$

Moreover, in air ($\mu = 1$),

$$H_x = -\frac{1}{\mu_0}\frac{\partial \phi}{\partial x} = \frac{1}{\mu_0}\frac{\partial A}{\partial y}, \tag{1.5.124a}$$

$$H_y = -\frac{1}{\mu_0}\frac{\partial \phi}{\partial y} = -\frac{1}{\mu_0}\frac{\partial A}{\partial x}. \tag{1.5.124b}$$

The Cauchy-Riemann equations are the necessary and sufficient conditions for a function to be analytic in Z-plane. For a function $F_w = u + i\ v$, these conditions are:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \tag{1.5.125a}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{1.5.126a}$$

In a medium free of magnetic material with $\mu = 1$, Eqs. (1.5.124) gives

$$\begin{aligned}\frac{\partial A}{\partial x} &= \frac{\partial \phi}{\partial y},\\ \frac{\partial A}{\partial y} &= -\frac{\partial \phi}{\partial x},\end{aligned}$$

which are the Cauchy-Riemann conditions for $W(z) = -(A + i\phi) + constant$ to be analytic. In the same way, B(z) (and similarly H(z)) is analytic if :

$$\begin{aligned}\frac{\partial B_y}{\partial x} &= \frac{\partial B_x}{\partial y},\\ \frac{\partial B_y}{\partial y} &= -\frac{\partial B_x}{\partial x},\end{aligned}$$

which are just Maxwell's equations in a current free region. It may be noted that the choice of variable $B(z)$ as $B(z) = B_y + iB_x$ is important since B_x and B_y do not the satisfy the Cauchy-Rieman conditions if the variable is $B_x + iB_y$.

Since $W(z)$ is analytic, the derivative of $W(z)$ gives the the complex field function :

$$\frac{dW}{dz} = -\frac{\partial A}{\partial x} - \ i \ \frac{\partial \phi}{\partial x} = i \ \frac{\partial A}{\partial y} - \frac{\partial \phi}{\partial y} = H_y + \ i \ H_x \ = \ H(z).$$

To deal with a region with current, a new analytic function is defined as follows :

$$F(z) = B(z) - \frac{1}{2}\mu_o J z^* = \left(B_y - \tfrac{1}{2}\mu_o Jx\right) \ + \ i \ \left(B_x + \tfrac{1}{2}\mu_o Jy\right), \tag{1.5.127}$$

where the current density J is constant throughout the region. The Cauchy-Riemann conditions become :

$$\begin{aligned}\frac{\partial B_y}{\partial x} - \tfrac{1}{2}\mu_o J &= \frac{\partial B_x}{\partial y} + \tfrac{1}{2}\mu_o J\\ \Longrightarrow \quad \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} &= \mu_o J\\ and, \quad \frac{\partial B_y}{\partial y} &= -\frac{\partial B_x}{\partial x}\\ \Longrightarrow \quad \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} &= 0\end{aligned}$$

which are Maxwell's equations in the presence of current.

1.5.4.1. Field due to an array of Line Currents

The complex potential at a point z, due a current flowing in a direction perpendicular to the Z-plane at $z = z_o$, is given by :

$$W(z) = \frac{I}{2\pi} \, log\,(z - z_o) + constant \;,$$

and the magnetic field is given by :

$$H(z) = \frac{dW}{dz} = \frac{I}{2\pi\,(z - z_o)}. \tag{1.5.128}$$

The direction of the field is that of $(z - z_o)^*$, which is perpendicular to the vector $(z - z_o)$. The superposition principle can be used to obtain the field due to n filaments with the k^{th} filament carrying I_k amperes and located at $z = z_k$:

$$H(z) = \sum_{k=1}^{n} \frac{I_k}{2\pi\,(z - z_k)}. \tag{1.5.129}$$

Cauchy's Residue Theorem gives [29]

$$\oint_C f(z) \; dz = 2\pi i \sum_k Res\,(a_k), \tag{1.5.130}$$

where $Res(a_k)$ are the residues which are defined as the coefficients of $\frac{1}{z - z_k}$ inside the contour C over which the contour integral of the function *f(z)* is taken. Applying this to Eq. (1.5.129) while taking the contour integral of the field around the wires in the Z-plane, one obtains

$$\oint H(z) \; dz = i \sum_{k=1}^{n} I_k. \tag{1.5.131}$$

which is basically Ampere's law.

The Cauchy integral formula [29] gives :

$$f(z_o) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_o)} dz, \tag{1.5.132}$$

where the function $f(z)$ is analytic everywhere within and on a closed contour C and $f(z_o)$ is the value of $f(z)$ at $z = z_o$.

1.5.4.2. Beth's Current Sheet Theorem

Beth's *"Current Sheet Theorem"* [12-18] can be derived from Eq. (1.5.131). As shown in Fig. 1.5.4 the current sheet is made up of a number of filaments carrying a total current ΔI perpendicular to the Z-plane along the curve from z to $z + \Delta z$. A contour integral on a closed path enclosing the current sheet will give

$$\oint H(z) \; dz \;=\; i \; \Delta I.$$

Now if the path is squeezed from the right and left sides (indicated by the subscripts R and L) on to the current sheet, then in the limiting case one obtains

$$H_R(z_o) \;-\; H_L(z_o) \;=\; i \; \frac{dI}{dz}, \tag{1.5.133}$$

where $H_R(z_o)$ and $H_L(z_o)$ are the limits of the analytic functions $H_R(z)$ and $H_L(z)$ when z approaches z_o from the right and left and $\frac{dI}{dz}$ is the limit of $\frac{\Delta I}{\Delta z}$ when Δz approaches 0 at any z.

The above equation Eqs. (1.5.133) is called Beth's current sheet theorem. To obtain another equation in potential form this equation is integrated to give

$$W_R(z_o) \;-\; W_L(z_o) \;=\; i \; I \;+\; Constant \;, \tag{1.5.134}$$

where $W_R(z_o)$ and $W_L(z_o)$ are the limits of the analytic functions $W_R(z)$ and $W_L(z)$ when z approaches z_o from the right and left.

Beth's Current Sheet

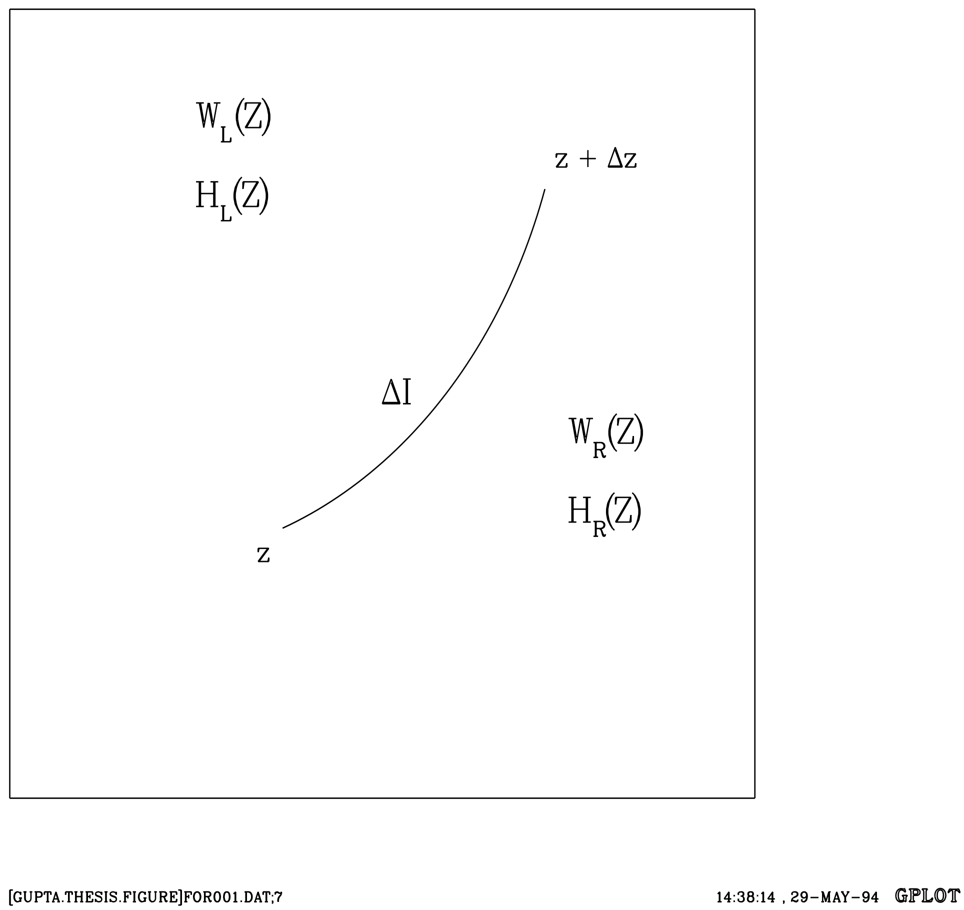

Figure 1.5.4: Beth's current sheet is shown here, which is made up of a number of filaments, carrying a total current ΔI perpendicular to the Z-plane along the curve from z to $z + \Delta z$. The sub-script "R" denotes the right side and "L" denotes the left side to the sheet.

1.5.4.3. Example – Cos(m θ) current distribution

As an example of use of the complex variable methods, expressions are derived here for the field due to a cylindrical current sheet at a radius $r = a$, as shown in Fig. 1.5.2. An angular current density distribution, mentioned earlier, is :

$$\frac{dI}{d\phi} = I_o \; cos\,(m\phi)\,.$$

In complex coordinates, the above current sheet is located at $z = a\; e^{i\phi}$. Then,

$$\frac{dI}{dz} = \left(\frac{dI}{d\phi}\right) / \left(\frac{dz}{d\phi}\right) = \frac{I_o \; cos\,(m\phi)}{i \; a \; e^{i\phi}}.$$

Since H(z) is analytic both inside and outside the current sheet, a general expression for the field to remain finite inside the current sheet $(r < a)$ is $H_{in} = \sum_n a_n z^n$ and for outside the current sheet $(r > a)$ is $H_{out} = \sum_n b_n z^{-n}$. To obtain the coefficients a_n and b_n, the fields (H_{in}) and (H_{out}) are linked using Beth's current sheet theorem (Eqs. (1.5.133)) as follows :

$$\begin{aligned} H_{out} \; - \; H_{in} &= I_o \frac{cos\,(m\phi)}{a \; e^{i\phi}} \\ &= \frac{I_o}{2 \; a} \left[e^{-i(m+1)\phi} + e^{i(m-1)\phi} \right] \\ &= \frac{I_o}{2 \; a} \left[\left(\frac{a}{z}\right)^{m+1} + \left(\frac{z}{a}\right)^{m-1} \right]. \end{aligned}$$

The right hand side of the above equation gives the field on the current sheet and it acts as a boundary condition which must match interior and exterior solutions. Hence a_n=0 for n$\neq$m-1 and b_n=0 for n$\neq$m+1, giving

$$\begin{aligned} H_{in} &= \frac{-I_o}{2 \; a} \left(\frac{z}{a}\right)^{m-1} \qquad & |z| < a, \\ H_{out} &= \frac{I_o}{2 \; a} \left(\frac{a}{z}\right)^{m+1} \qquad & |z| > a. \end{aligned}$$