Measurements of Field Quality Using Harmonic Coils

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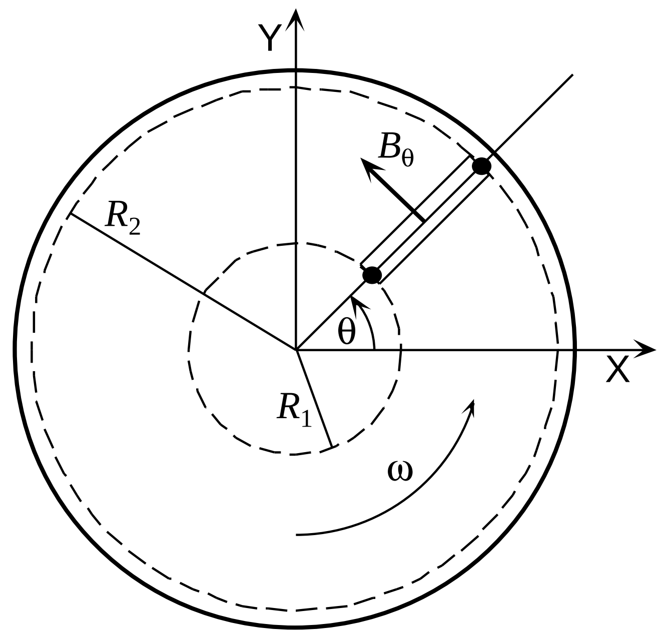
A Radial Coil (or B_θ Coil)

Number of Turns = N
Length of the coil = L



A radial coil has a flat loop of wire whose plane coincides with the radial plane of the rotating cylinder. The two sides of the loop are located at radii R_1 and R_2, as shown above. The flux through the coil at an angular orientation θ is:

$$\Phi(\theta) = NL\int_{R_1}^{R_2} B_\theta(r,\theta)\,dr = NL\int_{R_1}^{R_2}\sum_{n=1}^{\infty} C(n)\left(\frac{r}{R_{ref}}\right)^{n-1}\cos(n\theta - n\alpha_n)\,dr$$

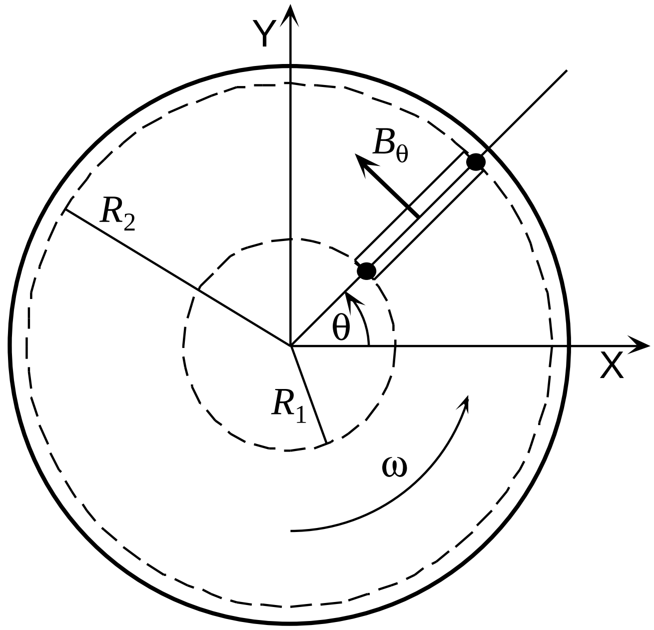
$$\Phi(\theta) = \sum_{n=1}^{\infty}\frac{NLR_{ref}}{n}\left[\left(\frac{R_2}{R_{ref}}\right)^n - \left(\frac{R_1}{R_{ref}}\right)^n\right]C(n)\cos(n\theta - n\alpha_n)$$

If the coil rotates with an angular velocity ω and $\theta = \delta$ is the angular position at time $t = 0$, then $\theta = \omega t + \delta$. The flux as a function of time is then:

$$\Phi(t) = \sum_{n=1}^{\infty}\frac{NLR_{ref}}{n}\left[\left(\frac{R_2}{R_{ref}}\right)^n - \left(\frac{R_1}{R_{ref}}\right)^n\right]C(n)\cos(n\omega t + n\delta - n\alpha_n)$$

A Radial Coil (or B_θ Coil)

Number of Turns = N
Length of the coil = L
Angular position at time t=0: δ



$$\Phi(\theta) = \sum_{n=1}^{\infty} \frac{NLR_{ref}}{n} \left[\left(\frac{R_2}{R_{ref}} \right)^n - \left(\frac{R_1}{R_{ref}} \right)^n \right] C(n) \cos(n\theta - n\alpha_n)$$

$$\Phi(t) = \sum_{n=1}^{\infty} \frac{NLR_{ref}}{n} \left[\left(\frac{R_2}{R_{ref}} \right)^n - \left(\frac{R_1}{R_{ref}} \right)^n \right] C(n) \cos(n\omega t + n\delta - n\alpha_n)$$

The voltage signal induced in the radial coil is:

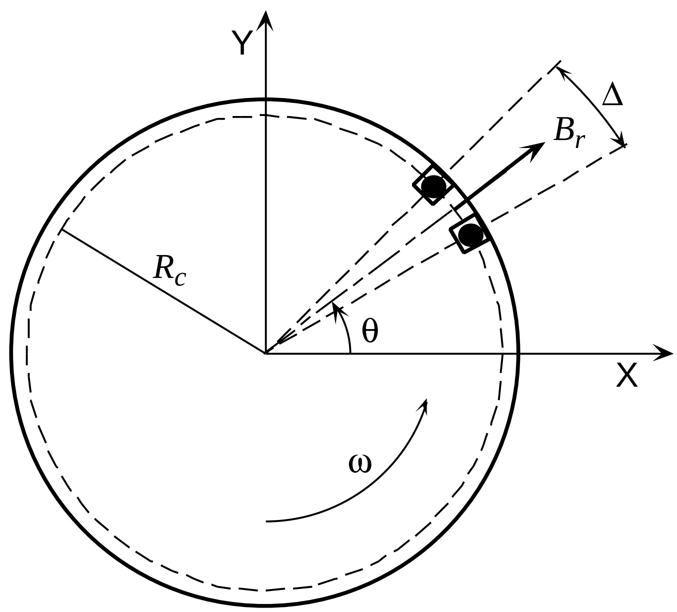
$$V(t) = -\left(\frac{\partial \Phi}{\partial t} \right) = \sum_{n=1}^{\infty} NLR_{ref}\, \omega \left[\left(\frac{R_2}{R_{ref}} \right)^n - \left(\frac{R_1}{R_{ref}} \right)^n \right] C(n) \sin(n\omega t + n\delta - n\alpha_n)$$

The amplitude of the voltage signal is proportional to the angular velocity. For analysis based on voltage signals, it is essential to control the angular velocity and make corrections for any speed fluctuations. The integrated voltage signal gives the flux, which is independent of angular velocity.

The above expressions assume that the two sides of the coil loop are located on the same side of the origin, as shown in the figure. If the two sides are located on opposite sides of the origin, as is true for many practical coils, then one should replace R_1 by $-R_1$ in the above equations.

A Tangential Coil (or B_r Coil)

Number of Turns = N
Length of the coil = L
Opening Angle = Δ



A tangential coil has a loop of wire whose plane is at right angles to the radius vector through the center of the loop. The two sides of the loop are both located at a radius of R_c, as shown above. The flux through the coil at an angular orientation θ is:

$$\Phi(\theta) = NL \int_{\theta-\Delta/2}^{\theta+\Delta/2} B_r(R_c,\theta)\, R_c d\theta = NL \int_{\theta-\Delta/2}^{\theta+\Delta/2} \sum_{n=1}^{\infty} C(n) \left(\frac{R_c}{R_{ref}}\right)^{n-1} \sin(n\theta - n\alpha_n)\, R_c d\theta$$

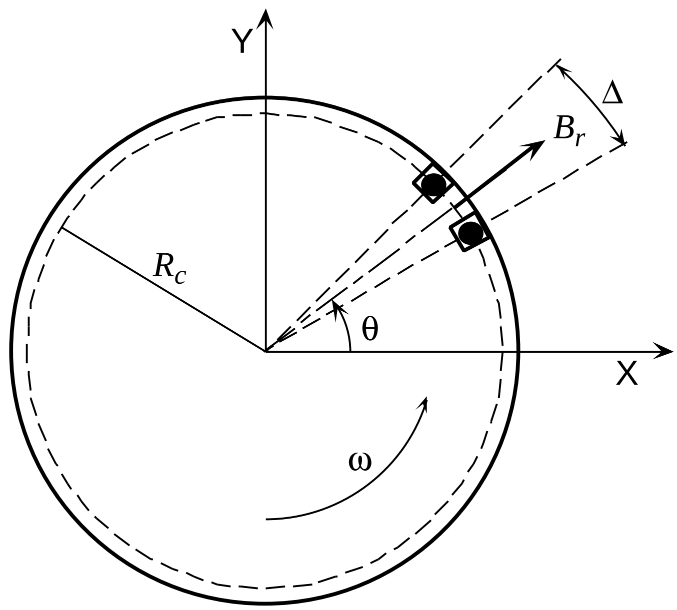
$$\Phi(\theta) = \sum_{n=1}^{\infty} \frac{2NLR_{ref}}{n} \left(\frac{R_c}{R_{ref}}\right)^{n} \sin\left(\frac{n\Delta}{2}\right) C(n) \sin(n\theta - n\alpha_n)$$

If the coil rotates with an angular velocity ω and $\theta = \delta$ is the angular position at time $t = 0$, then $\theta = \omega t + \delta$. The flux as a function of time is then:

$$\Phi(t) = \sum_{n=1}^{\infty} \frac{2NLR_{ref}}{n} \left(\frac{R_c}{R_{ref}}\right)^{n} \sin\left(\frac{n\Delta}{2}\right) C(n) \sin(n\omega t + n\delta - n\alpha_n)$$

A Tangential Coil (or B_r Coil)

Number of Turns = N
Length of the coil = L
Opening Angle = Δ
Angular position at time t=0: δ



$$\Phi(\theta) = \sum_{n=1}^{\infty} \frac{2NLR_{ref}}{n} \left(\frac{R_c}{R_{ref}}\right)^n \sin\left(\frac{n\Delta}{2}\right) C(n) \sin(n\theta - n\alpha_n)$$

$$\Phi(t) = \sum_{n=1}^{\infty} \frac{2NLR_{ref}}{n} \left(\frac{R_c}{R_{ref}}\right)^n \sin\left(\frac{n\Delta}{2}\right) C(n) \sin(n\omega t + n\delta - n\alpha_n)$$

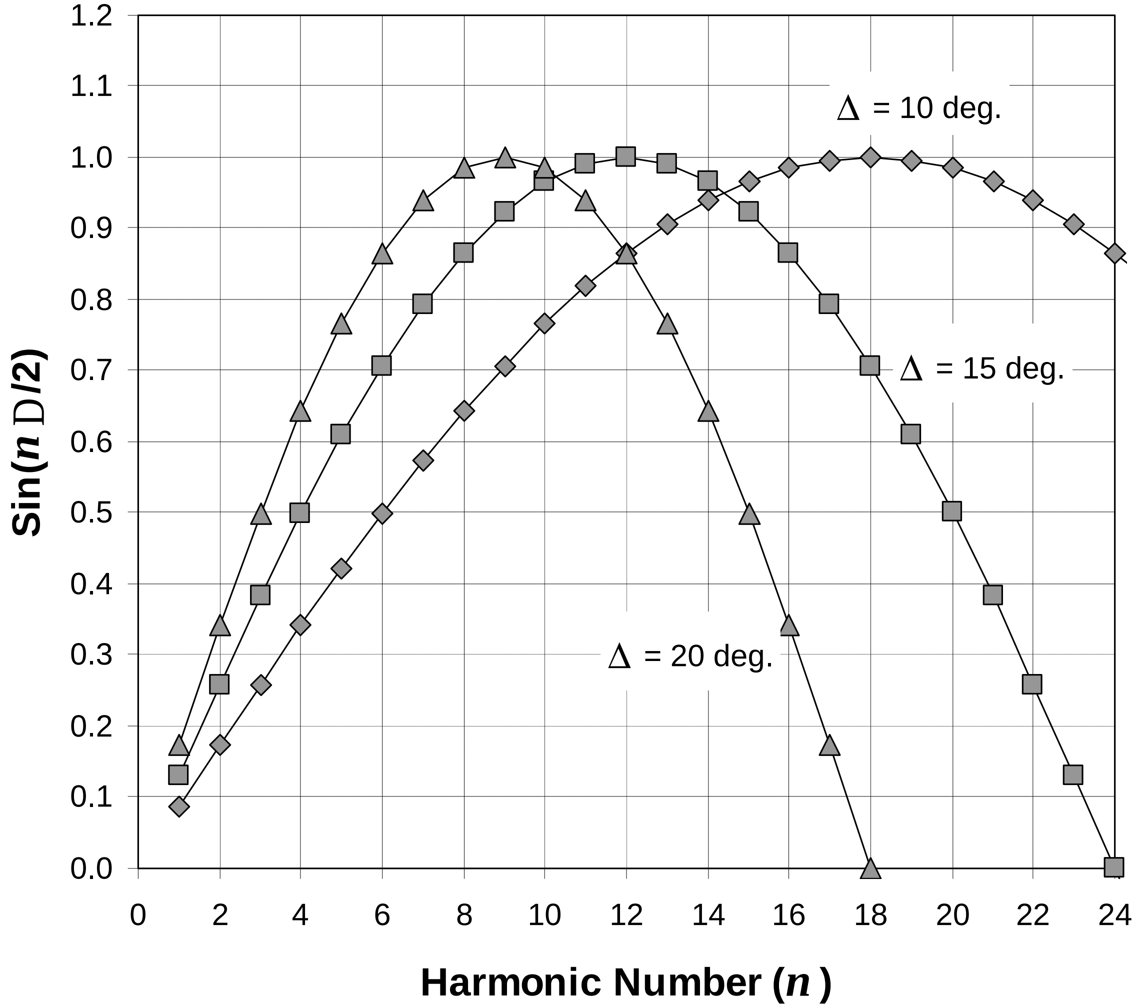
The voltage signal induced in the tangential coil is:

$$V(t) = -\left(\frac{\partial\Phi}{\partial t}\right) = -\sum_{n=1}^{\infty} 2NLR_{ref}\,\omega \left(\frac{R_c}{R_{ref}}\right)^n \sin\left(\frac{n\Delta}{2}\right) C(n) \cos(n\omega t + n\delta - n\alpha_n)$$

The amplitude of the voltage signal is proportional to the angular velocity. For analysis based on voltage signals, it is essential to control the angular velocity and make corrections for any speed fluctuations. The integrated voltage signal gives the flux, which is independent of angular velocity.

The radius, R_c, of the coil should be maximized to get good signal strength for higher harmonics. The opening angle, Δ, should be large enough to give enough signal and small enough so that $\sin(n\Delta/2)$ does not vanish for higher harmonics of interest ($\Delta << 2\pi/n_{max}$). Typically, Δ ~ 15 degrees.

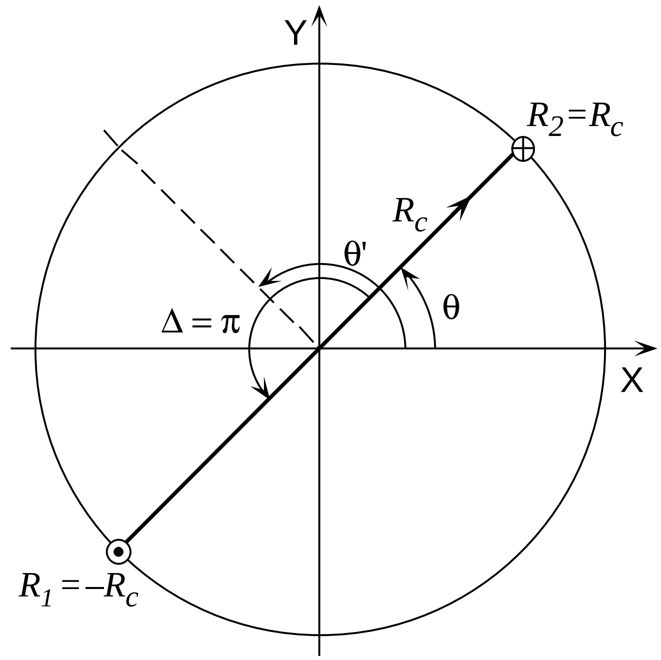
Sensitivity of a Tangential Coil to Various Harmonics ($R_c = R_{ref}$)



Typically, one is interested in precise measurement of about 15 harmonics. It is clear from the above plot that a tangential coil with an opening angle of 20 degrees rapidly loses sensitivity for higher harmonics, although it is more sensitive to lower harmonics as compared to a coil with 15 degrees opening angle. On the other hand, the sensitivity of a coil with an opening angle of only 10 degrees peaks at 36-pole term, which is of little relevance for accelerator physics. Such a narrow angle, therefore, sacrifices sensitivity for lower harmonics of interest with no useful outcome. It is clear from this plot that optimum value of the opening angle is $\Delta \sim 15$ degrees.

A "Dipole Coil" : Radial or Tangential ?

A "Dipole Coil" is a coil with a specific geometry which has the "Dipole Symmetry", namely an antisymmetry under rotation by π radians. The flux through this coil can be calculated by treating it as a *radial coil* with $R_1 = -R_c$ and $R_2 = +R_c$, oriented at an angle θ, as shown. The flux through the coil can also be calculated by treating it as a *tangential coil* with an opening angle of π radians, oriented at an angle of $\theta' = \theta + \pi/2$. Both approaches give the same result.



$$\Phi_{radial}(\theta) = \sum_{n=1}^{\infty} \frac{NLR_{ref}}{n}\left[\left(\frac{R_c}{R_{ref}}\right)^n - \left(\frac{-R_c}{R_{ref}}\right)^n\right] C(n)\cos(n\theta - n\alpha_n)$$

$$= \sum_{n=1}^{\infty} \frac{NLR_{ref}}{n}\left(\frac{R_c}{R_{ref}}\right)^n \left[1-(-1)^n\right] C(n)\cos(n\theta - n\alpha_n)$$

Similarly,

$$\Phi_{\mathrm{tang.}}(\theta') = \Phi_{\mathrm{tang.}}(\theta + \frac{\pi}{2}) = \sum_{n=1}^{\infty} \frac{2NLR_{ref}}{n}\left(\frac{R_c}{R_{ref}}\right)^n \sin\left(\frac{n\pi}{2}\right) C(n)\sin(n\theta + \frac{n\pi}{2} - n\alpha_n)$$

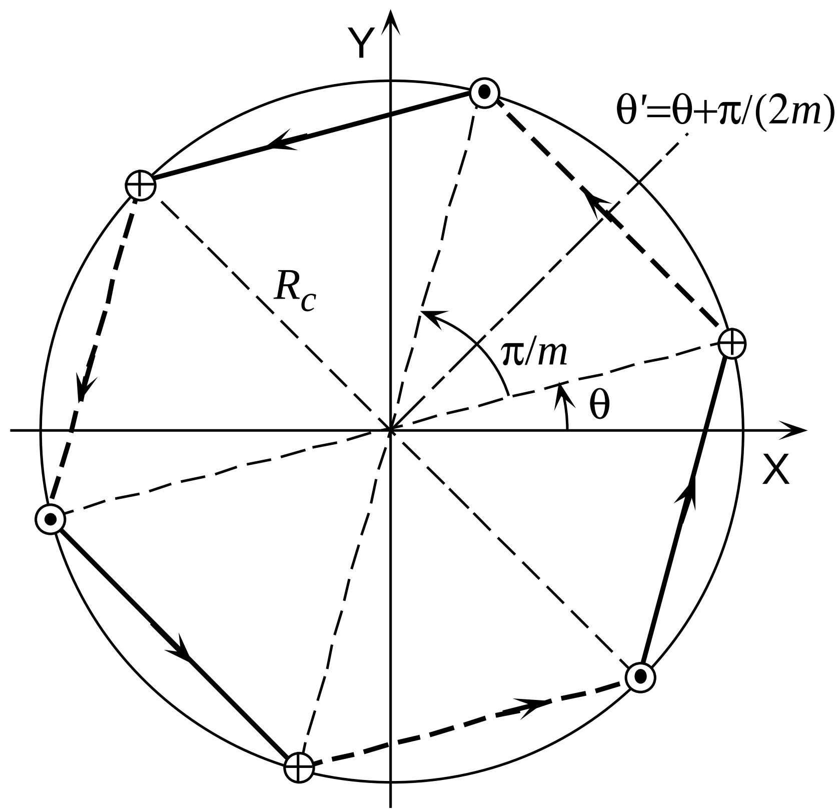
$$= \sum_{n=1}^{\infty} \frac{2NLR_{ref}}{n}\left(\frac{R_c}{R_{ref}}\right)^n \sin^2\left(\frac{n\pi}{2}\right) C(n)\cos(n\theta - n\alpha_n)$$

The terms with n = even vanish in both the expressions. The flux through a dipole coil is therefore given by:

$$\Phi_{Dipole}(\theta) = \sum_{\substack{n=1 \\ n=odd}}^{\infty} \frac{2NLR_{ref}}{n}\left(\frac{R_c}{R_{ref}}\right)^n C(n)\cos(n\theta - n\alpha_n)$$

A "Dipole Coil" is therefore sensitive to only the odd harmonics, i.e., dipole, sextupole, decapole, etc. Such a coil is almost universally used in both radial and tangential coil systems for "bucking" the main dipole field.

A Multipole Coil of Order *m* (A 2*m*-Pole Coil)



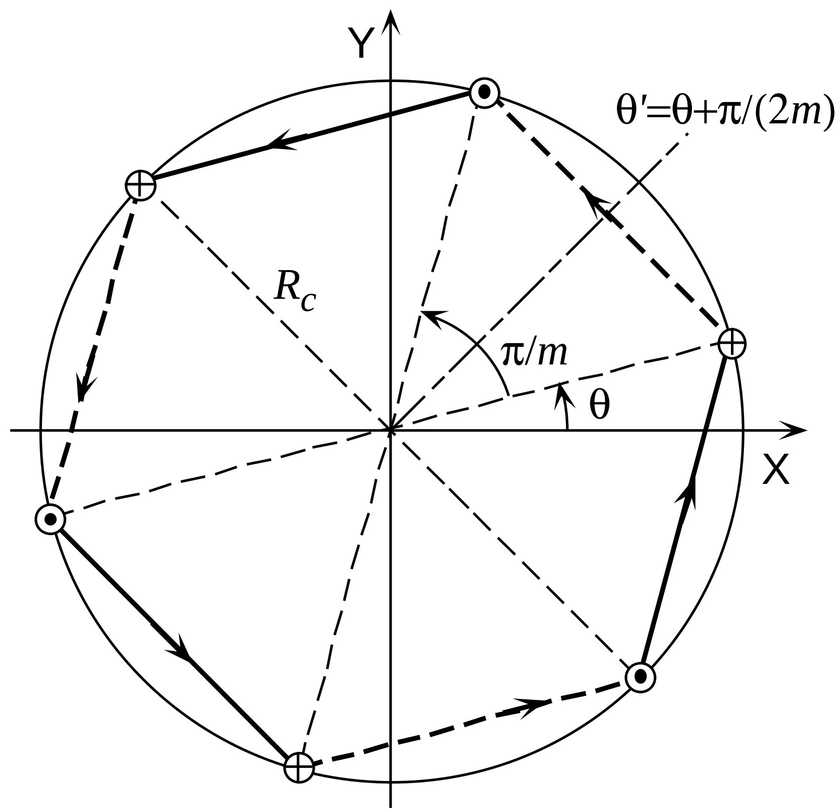
A multipole coil of order *m*, or a *2m*-pole coil is a coil with special geometry that has *m* loops connected in series, as shown in the figure. For any angular position characterized by the angle θ, the loops span the angular region of θ to $\theta+(\pi/m)$, $(\theta+2\pi/m)$ to $(\theta+3\pi/m)$, $(\theta+4\pi/m)$ to $(\theta+5\pi/m)$, and so on. The flux through such a coil as a function of θ can be easily calculated by treating it as an array of *m* identical tangential coils with opening angle of $\Delta = \pi/m$ and having angular positions of $\theta' = \theta+(\pi/2m)$, $\theta' + 2\pi/m$, $\theta' + 4\pi/m$, and so on.

The flux through the first segment of the coil is:

$$\Phi_1(\theta) = \Phi_{\text{tang.}}(\theta') = \sum_{n=1}^{\infty} \frac{2NLR_{ref}}{n} \left(\frac{R_c}{R_{ref}}\right)^n \sin\left(\frac{n\pi}{2m}\right) C(n) \sin(n\theta' - n\alpha_n)$$

$$= \operatorname{Im} \sum_{n=1}^{\infty} \frac{2NLR_{ref}}{n} \left(\frac{R_c}{R_{ref}}\right)^n \sin\left(\frac{n\pi}{2m}\right) C(n) e^{i(n\theta' - n\alpha_n)}$$

A Multipole Coil of Order *m* (A 2*m*-Pole Coil)



The flux through the first segment of the coil is:

$$\Phi_1(\theta) = \Phi_{\text{tang.}}(\theta') = \operatorname{Im}\sum_{n=1}^{\infty}\frac{2NLR_{ref}}{n}\left(\frac{R_c}{R_{ref}}\right)^n \sin\left(\frac{n\pi}{2m}\right) C(n)\, e^{i(n\theta' - n\alpha_n)}$$

The contribution due to harmonics which are EVEN MULTIPLES OF *m* VANISHES due to the sin($n\pi/2m$) factor. Let us now consider the total flux through the array of loops:

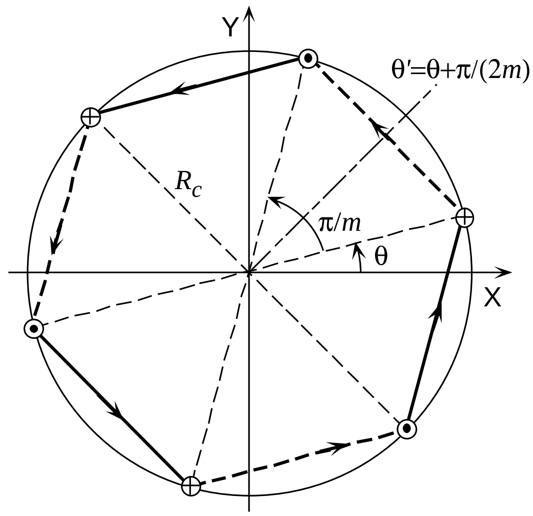
$$\Phi(\theta) = \operatorname{Im}\sum_{n=1}^{\infty}\boldsymbol{X}_n\, e^{in\theta'}\left[1 + e^{i2\pi n/m} + e^{i4\pi n/m} + \ldots (m \text{ terms})\right]$$

$$= \operatorname{Im}\sum_{n=1}^{\infty}\boldsymbol{X}_n\, e^{in\theta'}\left[\frac{1-\exp(i2n\pi)}{1-\exp(i2n\pi/m)}\right] = 0; \;\; \text{unless} \left(\frac{n}{m}\right) = \text{ integer}$$

where, $\boldsymbol{X}_n = \dfrac{2NLR_{ref}}{n}\left(\dfrac{R_c}{R_{ref}}\right)^n \sin\left(\dfrac{n\pi}{2m}\right) C(n) e^{-in\alpha_n}$ = 0 for (n/m)=even integer.

Therefore, all terms in the summation vanish, except for those values of *n* which are ODD MULTIPLES OF *m*.

A Multipole Coil of Order *m* (A 2*m*-Pole Coil)



$$\Phi(\theta) = \operatorname{Im}\sum_{n=1}^{\infty} \boldsymbol{X}_n e^{in\theta'}\left[\frac{1-\exp(i2n\pi)}{1-\exp(i2n\pi/m)}\right] = 0;\ \text{unless}\left(\frac{n}{m}\right) = \text{ integer}$$

where, $\boldsymbol{X}_n = \frac{2NLR_{ref}}{n}\left(\frac{R_c}{R_{ref}}\right)^n \sin\left(\frac{n\pi}{2m}\right) C(n) e^{-in\alpha_n}$ = 0 for (n/m)=even integer.

Therefore, all terms in the summation vanish, except for those values of n which are ODD MULTIPLES OF m. The total flux for the $2m$-pole coil can be written as:

$$\Phi(\theta) = \sum_{\substack{n=m \\ n=(2k+1)m}}^{\infty} \frac{2mNLR_{ref}}{n}\left(\frac{R_c}{R_{ref}}\right)^n C(n)\cos(n\theta - n\alpha_n)$$

where k is any integer, including zero. If the coil rotates with an angular velocity ω and $\theta = \delta$ is the initial angular position of the coil, then $\theta = \omega t + \delta$. The flux and the voltage at any time t are:

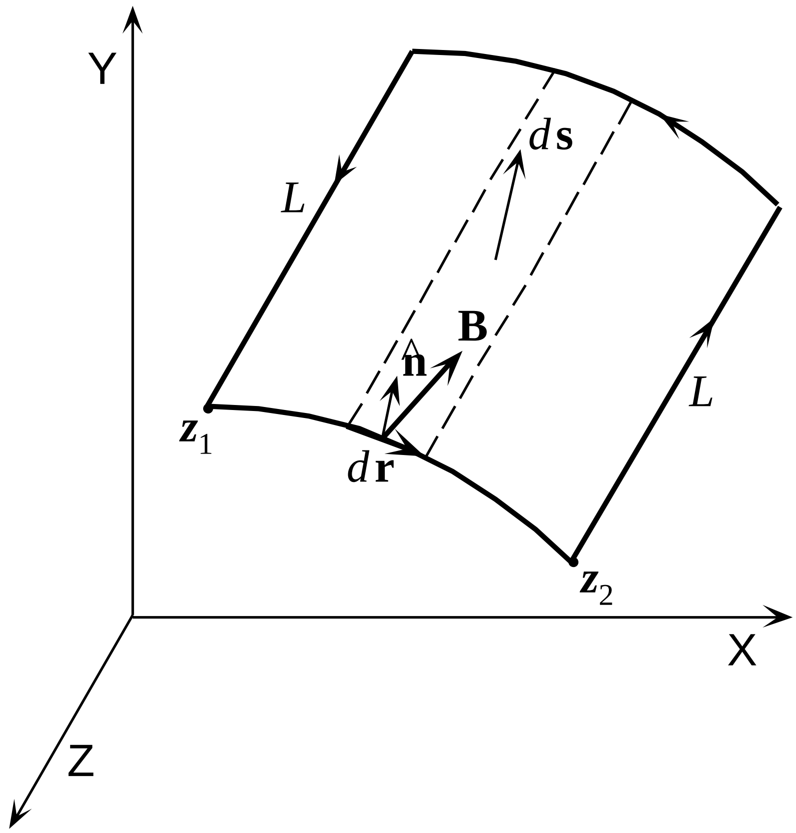
$$\Phi(t) = \sum_{\substack{n=m \\ n=(2k+1)m}}^{\infty} \frac{2mNLR_{ref}}{n}\left(\frac{R_c}{R_{ref}}\right)^n C(n)\cos(n\omega t + n\delta - n\alpha_n)$$

$$V(t) = -\frac{\partial\Phi(t)}{\partial t} = \sum_{\substack{n=m \\ n=(2k+1)m}}^{\infty} 2m\omega NLR_{ref}\left(\frac{R_c}{R_{ref}}\right)^n C(n)\sin(n\omega t + n\delta - n\alpha_n)$$

The results for a dipole coil are obtained by putting $m = 1$. Quadrupole coils ($m = 2$) are commonly used for bucking in tangential coil systems. Sextupole and other higher order coils may be used for special applications.

Flux Through a Coil of Arbitrary Shape

We consider a coil made up of a loop of wire running parallel to the *Z*-axis as shown in the figure. The shape of the coil is defined by a path from the point $\boldsymbol{z}_1$ to $\boldsymbol{z}_2$ in the complex plane. The length of the loop is L along the *negative* *Z*-axis, as shown.



$d\mathbf{r}$ = an element along the path from $\mathbf{z}_1$ to $\mathbf{z}_2$.

$d\mathbf{s} = \hat{\mathbf{n}}|d\mathbf{r}|L$ = element of area defined by the line element $d\mathbf{r}$

The flux through the area element $d\mathbf{s}$ is given by $\mathbf{B}.\ d\mathbf{s}$. The area element is given by the vector:

$$d\mathbf{s} = \hat{\mathbf{n}}|d\mathbf{r}|L = (\hat{\mathbf{z}}\times d\mathbf{r})L = \hat{\mathbf{z}}\times(\hat{\mathbf{x}}dx + \hat{\mathbf{y}}dy)L = (-dy\hat{\mathbf{x}} + dx\hat{\mathbf{y}})L$$

$$\therefore d\Phi = \mathbf{B}\cdot d\mathbf{s} = (B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}}).(-dy\hat{\mathbf{x}} + dx\hat{\mathbf{y}})L = (B_y dx - B_x dy)L$$

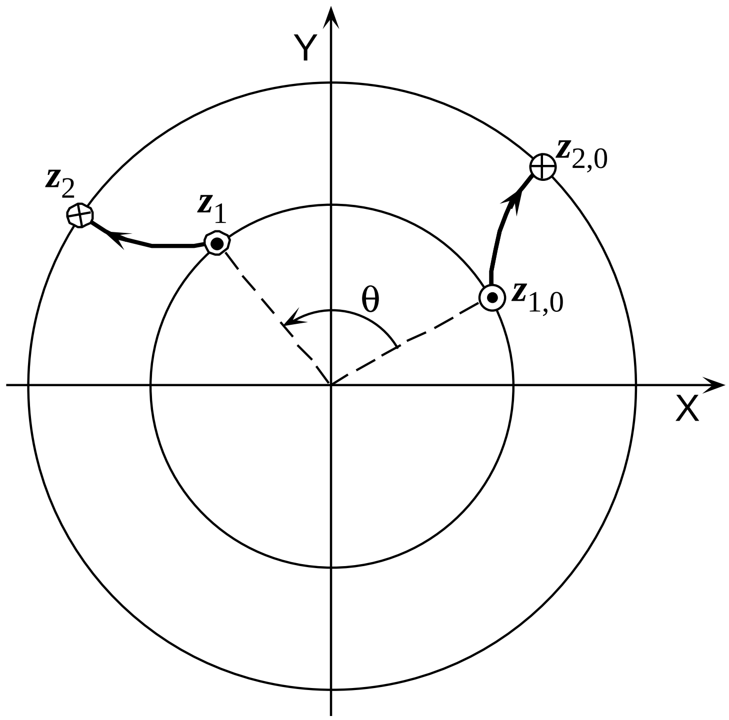
Let us evaluate the integral of the complex field $\boldsymbol{B}(\mathbf{z})$ from $\mathbf{z}_1$ to $\mathbf{z}_2$:

$$\int_{z_1}^{z_2} \boldsymbol{B}(\boldsymbol{z})d\boldsymbol{z} = \int_{z_1}^{z_2}(B_y + iB_x)(dx + idy) = \int(B_y dx - B_x dy) + i\int(B_x dx + B_y dy)$$

This leads us to the general result for a loop with N turns:

$$\Phi = NL\,\mathrm{Re}\left[\int_{z_1}^{z_2}\boldsymbol{B}(\boldsymbol{z})d\boldsymbol{z}\right] = \mathrm{Re}\left[\sum_{n=1}^{\infty}\left(\frac{NLR_{ref}}{n}\right)C(n)\exp(-in\alpha_n)\left\{\left(\frac{\boldsymbol{z}_2}{R_{ref}}\right)^n - \left(\frac{\boldsymbol{z}_1}{R_{ref}}\right)^n\right\}\right]$$

A Rotating Coil of Arbitrary Shape



Let us consider a rotating coil of arbitrary shape formed by a loop of wire passing through two points in the X-Y plane. In general, both the radial and the azimuthal coordinates of these two points will be different. The *radial coil* is a special case where the *azimuthal* coordinates of both the points are either the same, or differ by π. Similarly, the *tangential coil* is a special case where the *radial* coordinates of the two points are the same. Any angular position of the coil is characterized by an angle θ measured from an "initial position". If $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$ denote the locations of the two points in the complex plane at any instant, then the flux through the coil of length L and with N turns is:

$$\Phi(\theta) = \mathrm{Re}\left[\sum_{n=1}^{\infty}\left(\frac{NLR_{ref}}{n}\right)C(n)\exp(-in\alpha_n)\left\{\left(\frac{\boldsymbol{z}_2}{R_{ref}}\right)^n - \left(\frac{\boldsymbol{z}_1}{R_{ref}}\right)^n\right\}\right]$$

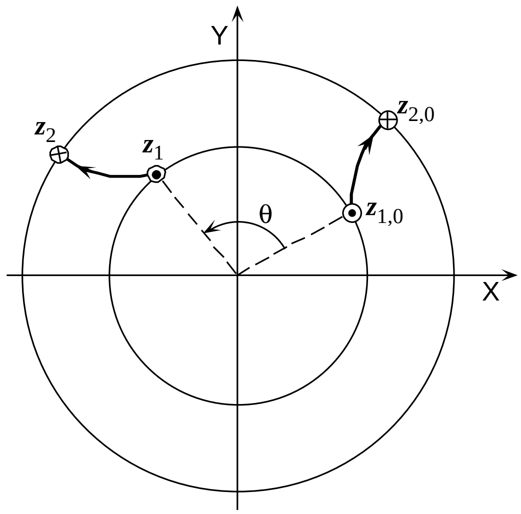
From the figure: $\boldsymbol{z}_1 = \boldsymbol{z}_{1,0}\exp(i\theta)$; $\boldsymbol{z}_2 = \boldsymbol{z}_{2,0}\exp(i\theta)$. Substituting in the above expression for the flux, we get:

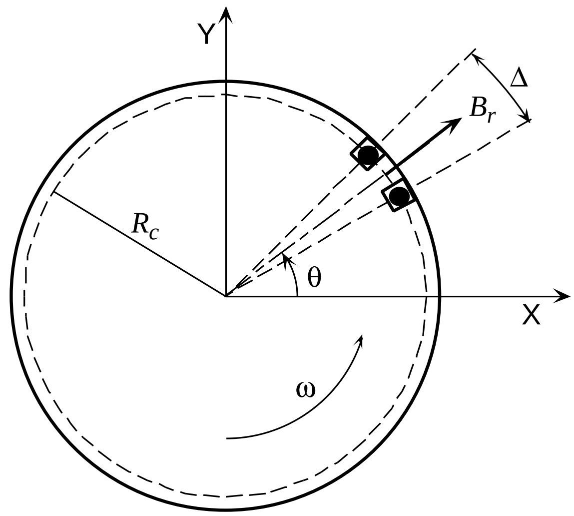
$$\Phi(\theta) = \mathrm{Re}\left[\sum_{n=1}^{\infty}\boldsymbol{K}_n\ \exp(in\theta)C(n)\exp(-in\alpha_n)\right]$$

where $\boldsymbol{K}_n$ is the "SENSITIVITY FACTOR" for the order n defined by:

$$\boldsymbol{K}_n = \left(\frac{NLR_{ref}}{n}\right)\left\{\left(\frac{\boldsymbol{z}_{2,0}}{R_{ref}}\right)^n - \left(\frac{\boldsymbol{z}_{1,0}}{R_{ref}}\right)^n\right\}$$

Tangential Coil as a Special Case of a General Coil





$$\Phi(\theta) = \mathrm{Re}\left[\sum_{n=1}^{\infty} \boldsymbol{K}_n \exp(in\theta) C(n) \exp(-in\alpha_n)\right]$$

$$\boldsymbol{K}_n = \left(\frac{NLR_{ref}}{n}\right)\left\{\left(\frac{\boldsymbol{z}_{2,0}}{R_{ref}}\right)^n - \left(\frac{\boldsymbol{z}_{1,0}}{R_{ref}}\right)^n\right\}$$

For a tangential coil, when θ is measured from the X-axis,

$$\boldsymbol{z}_{1,0} = R_c \exp(i\Delta / 2); \quad \boldsymbol{z}_{2,0} = R_c \exp(-i\Delta / 2)$$

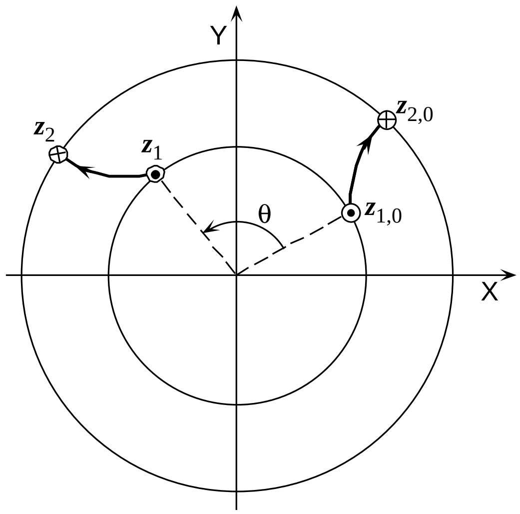
$$\therefore \boldsymbol{K}_n^{\text{tang.}} = -i \frac{2NLR_{ref}}{n}\left(\frac{R_c}{R_{ref}}\right)^n \sin\left(\frac{n\Delta}{2}\right)$$

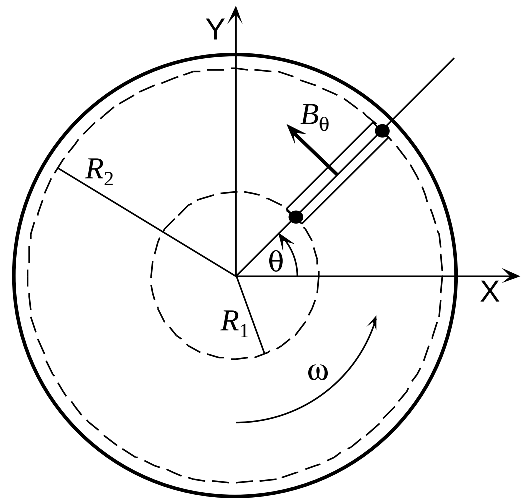
The Sensitivity Factor for a tangential coil is purely imaginary. The flux at the angular position θ is given by:

$$\Phi_{\text{tang}}(\theta) = \sum_{n=1}^{\infty} \frac{2NLR_{ref}}{n}\left(\frac{R_c}{R_{ref}}\right)^n \sin\left(\frac{n\Delta}{2}\right) C(n) \sin(n\theta - n\alpha_n)$$

which is the same expression as derived by directly integrating the radial component, $B_r(R_c,\theta)$, over the angular extent of the coil.

Radial Coil as a Special Case of a General Coil





$$\Phi(\theta) = \mathrm{Re}\left[\sum_{n=1}^{\infty} \boldsymbol{K}_n \ \exp(in\theta) C(n) \exp(-in\alpha_n)\right]$$

$$\boldsymbol{K}_n = \left(\frac{NLR_{ref}}{n}\right)\left\{\left(\frac{\boldsymbol{z}_{2,0}}{R_{ref}}\right)^n - \left(\frac{\boldsymbol{z}_{1,0}}{R_{ref}}\right)^n\right\}$$

For a radial coil, when θ is measured from the X-axis, $\boldsymbol{z}_{1,0} = \pm R_1; \quad \boldsymbol{z}_{2,0} = R_2$. It should be noted that $\boldsymbol{z}_{1,0} = +R_1$ when R_1 and R_2 are on the same side of the center and $\boldsymbol{z}_{1,0} = -R_1$ when R_1 and R_2 are on the opposite sides of the center.

$$\therefore \boldsymbol{K}_n^{\text{radial}} = \frac{NLR_{ref}}{n}\left[\left(\frac{R_2}{R_{ref}}\right)^n - \left(\frac{\pm R_1}{R_{ref}}\right)^n\right]$$

The Sensitivity Factor for a radial coil is purely real. The flux at the angular position θ is given by:

$$\Phi_{\text{radial}}(\theta) = \sum_{n=1}^{\infty} \frac{NLR_{ref}}{n}\left[\left(\frac{R_2}{R_{ref}}\right)^n - \left(\frac{\pm R_1}{R_{ref}}\right)^n\right] C(n)\cos(n\theta - n\alpha_n)$$

which is the same expression as derived by directly integrating the azimuthal component, $B_\theta(r,\theta)$, over the radial extent of the coil.

An Array of Rotating Coils

Let us consider an array of M coils mounted on the same rotating system. Let the sensitivity factor of the j-th coil for the n-th harmonic be denoted by $\boldsymbol{K}_n^{(j)},\ j = 1,2,3,\cdots M$. Let all these coils be connected either in series, or in opposition, to generate a combined signal. The total flux through this array of coils is the algebraic sum of the fluxes through individual coils:

$$\Phi(\theta) = \sum_{j=1}^{M} \varepsilon_j \Phi_j(\theta)$$

where $\varepsilon_j = +1$ if the j-th coil is connected in series, and $\varepsilon_j = -1$ if the j-th coil is connected in opposition. From the general formula for the flux through an individual coil, we obtain:

$$\begin{aligned}
\Phi(\theta) &= \sum_{j=1}^{M} \varepsilon_j \left[\mathrm{Re} \sum_{n=1}^{\infty} \boldsymbol{K}_n^{(j)} C(n) \exp(-in\alpha_n) \exp(in\theta) \right] \\
&= \mathrm{Re} \left[\sum_{j=1}^{M} \sum_{n=1}^{\infty} \varepsilon_j \boldsymbol{K}_n^{(j)} C(n) \exp(-in\alpha_n) \exp(in\theta) \right]; \text{ since } \varepsilon_j \text{ is real.} \\
&= \mathrm{Re} \sum_{n=1}^{\infty} \left(\sum_{j=1}^{M} \varepsilon_j \boldsymbol{K}_n^{(j)} \right) C(n) \exp(-in\alpha_n) \exp(in\theta) \\
&= \mathrm{Re} \sum_{n=1}^{\infty} \boldsymbol{K}_n \ C(n) \exp(-in\alpha_n) \exp(in\theta)
\end{aligned}$$

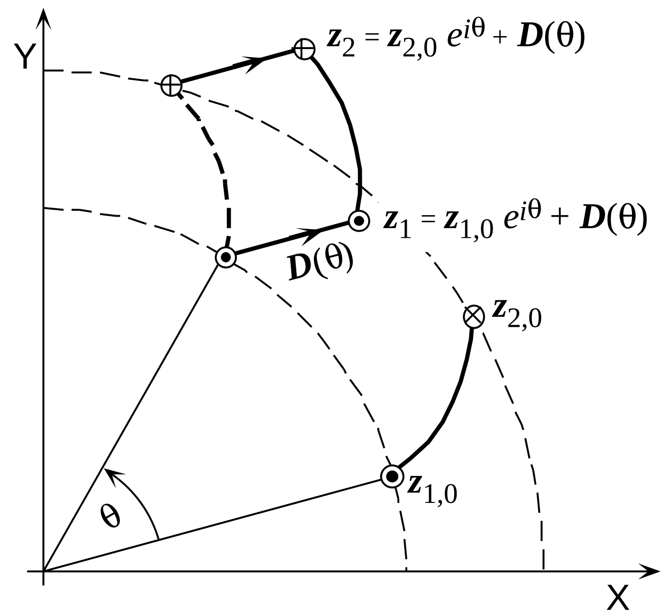
where $\boldsymbol{K}_n$ is the overall sensitivity of the array. From the above equation, it is clear that the sensitivity of an array of M coils connected either in series or in opposition is given by an algebraic sum of the sensitivities of the individual coils:

$$\boldsymbol{K}_n = \sum_{j=1}^{M} \varepsilon_j \boldsymbol{K}_n^{(j)}$$

If the individual coils are properly designed and ε_j are appropriately chosen, the overall sensitivity of an array of coils to a particular harmonic (or several harmonics) can be made zero. This is the principle used in “bucking”.

Transverse Vibrations of the Rotation Axis

Let us consider a rotating coil of a general shape whose rotation axis has a displacement as the coil rotates. This displacement, $\boldsymbol{D}(\theta)$, of the rotation axis in the complex $\boldsymbol{z}$-plane may be a function of the azimuthal angle, θ. The positions of the two sides of the coil loop at any angular position, θ, are given by:



$$\boldsymbol{z}_1 = \boldsymbol{z}_{1,0}\exp(i\theta) + \boldsymbol{D}(\theta); \quad \boldsymbol{z}_2 = \boldsymbol{z}_{2,0}\exp(i\theta) + \boldsymbol{D}(\theta)$$

The expression for flux at any angular position, θ, is:

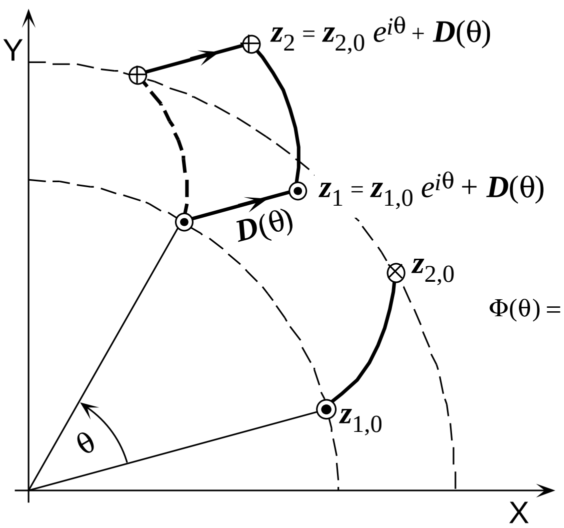
$$\Phi(\theta) = \mathrm{Re}\left[\sum_{n=1}^{\infty}\left(\frac{NLR_{ref}}{n}\right)C(n)\exp(-in\alpha_n)\left\{\left(\frac{\boldsymbol{z}_2}{R_{ref}}\right)^n - \left(\frac{\boldsymbol{z}_1}{R_{ref}}\right)^n\right\}\right]$$

$$\boldsymbol{K}_n = \left(\frac{NLR_{ref}}{n}\right)\left\{\left(\frac{\boldsymbol{z}_{2,0}}{R_{ref}}\right)^n - \left(\frac{\boldsymbol{z}_{1,0}}{R_{ref}}\right)^n\right\}$$ SENSITIVITY FACTOR FOR *n-th* HARMONIC

Case I: A Pure Dipole Field:

For a pure dipole field ($n = 1$), the expression for flux involves only the quantity $\boldsymbol{z}_2 - \boldsymbol{z}_1 = [\boldsymbol{z}_{2,0} - \boldsymbol{z}_{1,0}].\exp(i\theta)$, which is independent of the displacement, $\boldsymbol{D}(\theta)$. Thus, the flux linked through a coil in a pure dipole field is unaffected by transverse displacements of the rotation axis. This result is not too surprising because displacements in a pure dipole field do not produce any feed down harmonics.

Transverse Vibrations of the Rotation Axis



$$\boldsymbol{z}_1 = \boldsymbol{z}_{1,0} \exp(i\theta) + \boldsymbol{D}(\theta)$$

$$\boldsymbol{z}_2 = \boldsymbol{z}_{2,0} \exp(i\theta) + \boldsymbol{D}(\theta)$$

$$\Phi(\theta) = \mathrm{Re}\left[\sum_{n=1}^{\infty}\left(\frac{NLR_{ref}}{n}\right)C(n)\exp(-in\alpha_n)\left\{\left(\frac{\boldsymbol{z}_2}{R_{ref}}\right)^n - \left(\frac{\boldsymbol{z}_1}{R_{ref}}\right)^n\right\}\right]$$

$$\boldsymbol{K}_n = \left(\frac{NLR_{ref}}{n}\right)\left\{\left(\frac{\boldsymbol{z}_{2,0}}{R_{ref}}\right)^n - \left(\frac{\boldsymbol{z}_{1,0}}{R_{ref}}\right)^n\right\}$$

Case II: A Pure 2n-Pole Field:

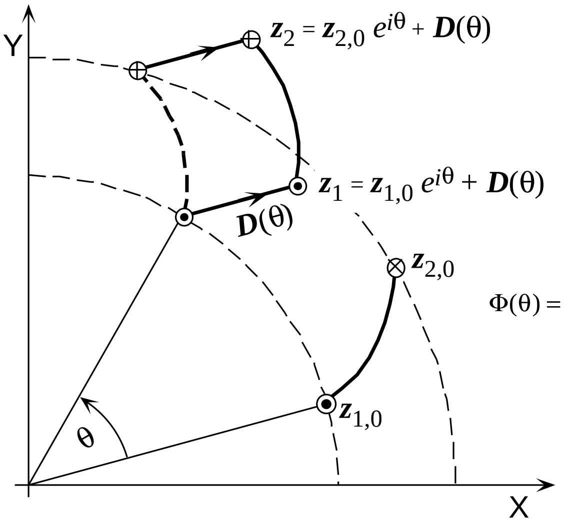
An approximate expression for the flux in a pure $2n$-pole field can be evaluated by using a binomial expansion and neglecting terms of the second and higher order in $[\boldsymbol{D}(\theta)/R_{ref}]$, assuming that $|\boldsymbol{D}(\theta)| << R_{ref}$. We get:

$$\Phi_n(\theta) \approx \mathrm{Re}\left[\boldsymbol{K}_n e^{in\theta}\, C(n) e^{-in\alpha_n}\right] + \mathrm{Re}\left[\boldsymbol{K}_{n-1} e^{i(n-1)\theta}\left\{\frac{(n-1)\boldsymbol{D}(\theta)}{R_{ref}}\right\} C(n) e^{-in\alpha_n}\right] + \cdots$$

The flux picked up by a rotating coil in a pure $2n$-pole field is, in general, affected by transverse displacements of the rotation axis. To a first approximation, the effect on the flux is proportional to the amplitude of the displacement and the sensitivity of the coil to the $2(n–1)$-pole terms. It should be noted that the highest power of $\boldsymbol{D}(\theta)$ in the expression for the flux from a $2n$-pole field is $(n-1)$. Thus, only the first term in the above expression survives for a pure dipole field, whereas the first two terms represent the complete expression for flux in a pure quadrupole field. For fields of higher multipolarities, other higher order terms are also present, but can be neglected in practice if the condition $|\boldsymbol{D}(\theta)| << R_{ref}$ is satisfied.

If the coil is replaced by an array of coils whose combined sensitivity to the $2(n-1)$-pole term is zero, then the effect of small transverse vibrations is practically eliminated. **This is the basis for bucking the $(n-1)$th harmonic.**

Periodic Transverse Motion of the Rotation Axis



$$\boldsymbol{z}_1 = \boldsymbol{z}_{1,0}\exp(i\theta) + \boldsymbol{D}(\theta)$$

$$\boldsymbol{z}_2 = \boldsymbol{z}_{2,0}\exp(i\theta) + \boldsymbol{D}(\theta)$$

$$\Phi(\theta) = \mathrm{Re}\left[\sum_{n=1}^{\infty}\left(\frac{NLR_{ref}}{n}\right)C(n)\exp(-in\alpha_n)\left\{\left(\frac{\boldsymbol{z}_2}{R_{ref}}\right)^n - \left(\frac{\boldsymbol{z}_1}{R_{ref}}\right)^n\right\}\right]$$

$$\boldsymbol{K}_n = \left(\frac{NLR_{ref}}{n}\right)\left\{\left(\frac{\boldsymbol{z}_{2,0}}{R_{ref}}\right)^n - \left(\frac{\boldsymbol{z}_{1,0}}{R_{ref}}\right)^n\right\}$$

If the displacement amplitude, $\boldsymbol{D}(\theta)$, is a periodic function of θ :

$$\boldsymbol{D}(\theta) = \sum_{p=-\infty}^{\infty}\boldsymbol{D}_p\exp(ip\theta)$$

In a pure $2n$-pole field:

$$\Phi_n(\theta) \approx \mathrm{Re}\left[\boldsymbol{K}_n e^{in\theta}\,C(n)e^{-in\alpha_n}\right] + \mathrm{Re}\left[\boldsymbol{K}_{n-1}e^{i(n-1)\theta}\left\{\frac{(n-1)\boldsymbol{D}(\theta)}{R_{ref}}\right\}C(n)e^{-in\alpha_n}\right] + \cdots$$

$$\boldsymbol{D}(\theta)e^{i(n-1)\theta} = \sum_{p=-n+2}^{\infty}\boldsymbol{D}_p e^{i(p+n-1)\theta} + \boldsymbol{D}_{-n+1} + \sum_{p=n}^{\infty}\boldsymbol{D}_{-p}e^{-i(p-n+1)\theta}$$

$$\mathrm{Re}\sum_{p=n}^{\infty}\boldsymbol{D}_{-p}e^{-i(p-n+1)\theta}\boldsymbol{K}_{n-1}C(n)e^{-in\alpha_n} = \mathrm{Re}\sum_{p=n}^{\infty}\boldsymbol{D}^{*}_{-p}e^{i(p-n+1)\theta}\boldsymbol{K}^{*}_{n\text{-}1}C(n)e^{in\alpha_n}$$

$$\Phi_n(q) \approx \mathrm{Re}\left[\boldsymbol{K}_n e^{inq}C(n)e^{-ina_n}\right] + \mathrm{Re}\left[\sum_{p=-n+2}^{\infty}\boldsymbol{K}_{n-1}e^{i(p+n-1)q}\,\frac{(n-1)\boldsymbol{D}_p}{R_{ref}}C(n)e^{-ina_n}\right]$$

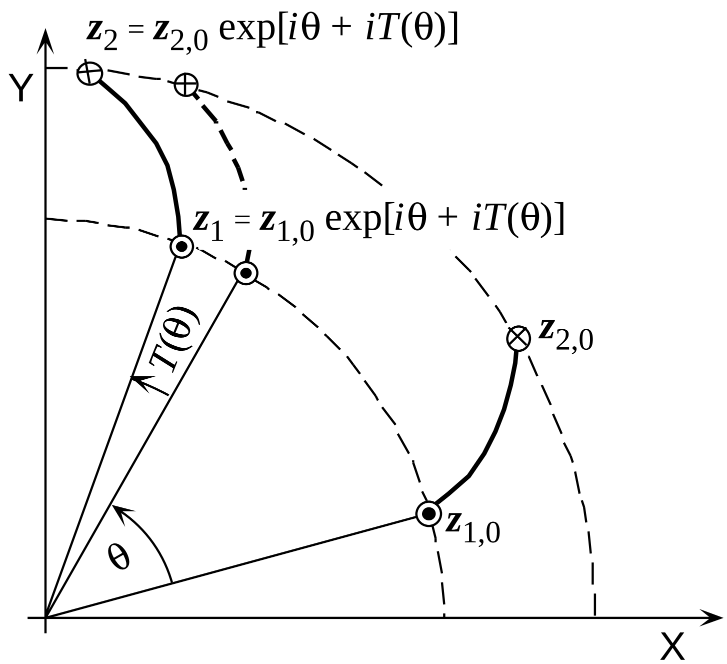
$$+\mathrm{Re}\left[\boldsymbol{K}_{n-1}\frac{(n-1)\boldsymbol{D}_{-n+1}}{R_{ref}}C(n)e^{-ina_n}\right] + \mathrm{Re}\left[\sum_{p=n}^{\infty}\boldsymbol{K}^{*}_{n-1}e^{i(p-n+1)q}\,\frac{(n-1)\boldsymbol{D}^{*}_{-p}}{R_{ref}}C(n)e^{ina_n}\right]$$

The amount of “Spurious” $2m$-pole harmonics in the measured flux is:

$$C'(m)e^{-ima'_m} \approx (n-1)\left[\left(\frac{\boldsymbol{K}_{n-1}}{\boldsymbol{K}_m}\right)\left(\frac{\boldsymbol{D}_{m-(n-1)}}{R_{ref}}\right)C(n)e^{-ina_n} + \left(\frac{\boldsymbol{K}^{*}_{n-1}}{\boldsymbol{K}_m}\right)\left(\frac{\boldsymbol{D}^{*}_{-m-(n-1)}}{R_{ref}}\right)C(n)e^{ina_n}\right]$$

For a sin(pθ) displacement, possible values of m = (p+n–1), (p–n+1), (n–p–1)

Torsional Vibrations of the Rotation Axis



Let us consider a type of rotational imperfection where the position of the rotating coil at angular postion θ is not at θ, but at an angle of $\theta + T(\theta)$, as shown in the figure. Such an imperfection may result either from an actual torsional vibration of the rotating coil, or it could be due to errors in the triggering of the data acquision. In general, the angular shift is a function of the angle. The position of the coil is characterized by:

$$\boldsymbol{z}_1 = \boldsymbol{z}_{1,0}\exp[i\theta + iT(\theta)]; \quad \boldsymbol{z}_2 = \boldsymbol{z}_{2,0}\exp[i\theta + iT(\theta)]$$

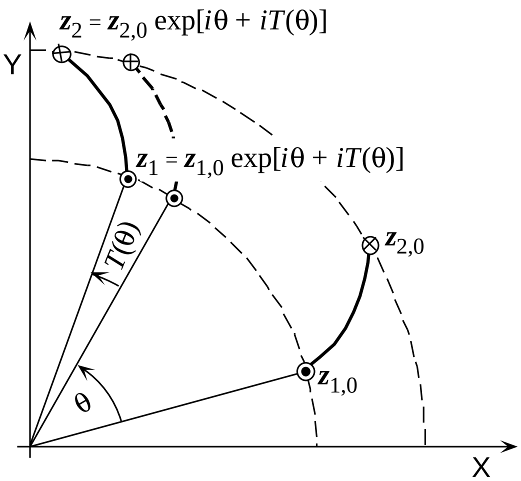
The flux through the coil is given by:

$$\Phi(\theta) = \mathrm{Re}\left[\sum_{n=1}^{\infty}\left(\frac{NLR_{ref}}{n}\right)C(n)\exp(-in\alpha_n)\left\{\left(\frac{\boldsymbol{z}_2}{R_{ref}}\right)^n - \left(\frac{\boldsymbol{z}_1}{R_{ref}}\right)^n\right\}\right]$$

The sensitivity of the “perfect coil” to the $2n$-pole field is defined as:

$$\boldsymbol{K}_n = \left(\frac{NLR_{ref}}{n}\right)\left\{\left(\frac{\boldsymbol{z}_{2,0}}{R_{ref}}\right)^n - \left(\frac{\boldsymbol{z}_{1,0}}{R_{ref}}\right)^n\right\}$$

Torsional Vibrations of the Rotation Axis



$$\Phi(\theta) = \mathrm{Re}\left[\sum_{n=1}^{\infty}\left(\frac{NLR_{ref}}{n}\right)C(n)\exp(-in\alpha_n)\left\{\left(\frac{z_2}{R_{ref}}\right)^n - \left(\frac{z_1}{R_{ref}}\right)^n\right\}\right]$$

$$\boldsymbol{K}_n = \left(\frac{NLR_{ref}}{n}\right)\left\{\left(\frac{z_{2,0}}{R_{ref}}\right)^n - \left(\frac{z_{1,0}}{R_{ref}}\right)^n\right\}$$

$$z_1 = z_{1,0}\exp[i\theta + iT(\theta)]$$
$$z_2 = z_{2,0}\exp[i\theta + iT(\theta)]$$

$$\Phi(\theta) = \mathrm{Re}\left[\sum_{n=1}^{\infty}\boldsymbol{K}_n\exp(in\theta).\exp\{inT(\theta)\}C(n)\exp(-in\alpha_n)\right]$$

In practice, the angular error, $T(\theta)$, is expected to be very small. Therefore,

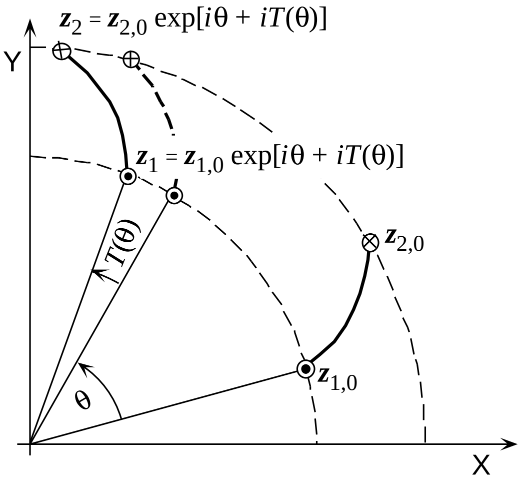
$$\exp\{inT(\theta)\} \approx 1 + inT(\theta)$$

$$\Phi(\theta) \approx \mathrm{Re}\left[\sum_{n=1}^{\infty}\boldsymbol{K}_n e^{in\theta}.C(n)e^{-in\alpha_n}\right] + \mathrm{Re}\left[\sum_{n=1}^{\infty} inT(\theta)\boldsymbol{K}_n e^{in\theta}.C(n)e^{-in\alpha_n}\right]$$

To a good approximation, the amplitude of distortion in a given harmonic component of the flux seen by the coil is proportional to the amplitude of the distortion as well as the sensitivity of the coil to the $2n$-pole terms.

If the magnet has only one dominant harmonic, then the effect of torsional vibrations can be minimized by making the sensitivity of the coil (or an array of coils) zero for that particular hamonic. **This is the basis for bucking out the dominant harmonic term from the pick up signal.** It should be noted that if the magnet has large allowed or unallowed multipoles, the effect of torsional vibrations is not completely cancelled.

Torsional Vibrations of the Rotation Axis: Periodic Displacements



$$\Phi(\theta) \approx \mathrm{Re}\left[\sum_{n=1}^{\infty} \boldsymbol{K}_n e^{in\theta}.C(n)e^{-in\alpha_n}\right] + \mathrm{Re}\left[\sum_{n=1}^{\infty} inT(\theta)\boldsymbol{K}_n e^{in\theta}.C(n)e^{-in\alpha_n}\right]$$

$$\boldsymbol{K}_n = \left(\frac{NLR_{ref}}{n}\right)\left\{\left(\frac{\boldsymbol{z}_{2,0}}{R_{ref}}\right)^n - \left(\frac{\boldsymbol{z}_{1,0}}{R_{ref}}\right)^n\right\}$$

If the displacement amplitude, $T(\theta)$, is a periodic function of θ :

$$T(\theta) = \sum_{p=-\infty}^{\infty} \boldsymbol{T}_p \exp(ip\theta)$$

$$T(\theta)e^{in\theta} = \sum_{p=-(n-1)}^{\infty} \boldsymbol{T}_p e^{i(p+n)\theta} + \boldsymbol{T}_{-n} + \sum_{p=n+1}^{\infty} \boldsymbol{T}_{-p} e^{-i(p-n)\theta}$$

$$\mathrm{Re}\left[in\boldsymbol{T}_{-p}\boldsymbol{K}_n e^{-i(p-n)\theta}C(n)e^{-in\alpha_n}\right] = \mathrm{Re}\left[-in\boldsymbol{T}^*_{-p}\boldsymbol{K}^*_n e^{i(p-n)\theta}C(n)e^{in\alpha_n}\right]$$

In a pure $2n$-pole field:

$$\Phi_n(\theta) \approx \mathrm{Re}\left[\boldsymbol{K}_n e^{in\theta}C(n)e^{-in\alpha_n}\right] + \mathrm{Re}\left[\sum_{p=-(n-1)}^{\infty} in\boldsymbol{T}_p\boldsymbol{K}_n e^{i(p+n)\theta}C(n)e^{-in\alpha_n}\right]$$
$$+\mathrm{Re}\left[in\boldsymbol{T}_{-n}\boldsymbol{K}_n C(n)e^{-in\alpha_n}\right] + \mathrm{Re}\left[\sum_{p=n+1}^{\infty} -in\boldsymbol{T}^*_{-p}\boldsymbol{K}^*_n e^{i(p-n)\theta}C(n)e^{in\alpha_n}\right]$$

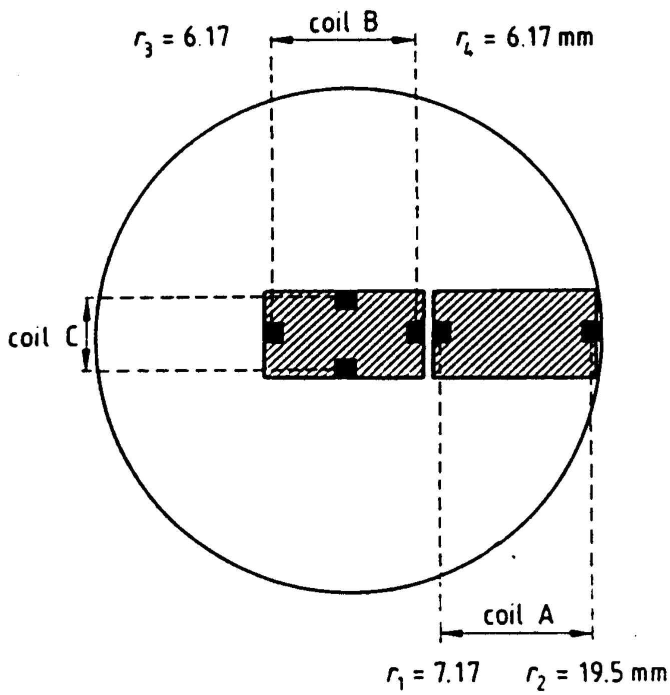
The amount of "Spurious" $2m$-pole harmonics in the measured flux is:

$$C'(m)e^{-im\alpha'_m} \approx in\left[\left(\frac{\boldsymbol{K}_n}{\boldsymbol{K}_m}\right)\boldsymbol{T}_{m-n}C(n)e^{-in\alpha_n} - \left(\frac{\boldsymbol{K}^*_n}{\boldsymbol{K}_m}\right)\boldsymbol{T}^*_{-m-n}C(n)e^{in\alpha_n}\right]$$

If $T(\theta)$ has a simple angular dependence of the form $T\cos(p\theta)$ or $T\sin(p\theta)$ then it will produce spurious harmonics corresponding to $2(n+p)$-pole and $2|(n-p)|$-pole field.

Example of a Practical Radial Coil with Bucking

HERA Dipole Coil



Coil A: Main Coil

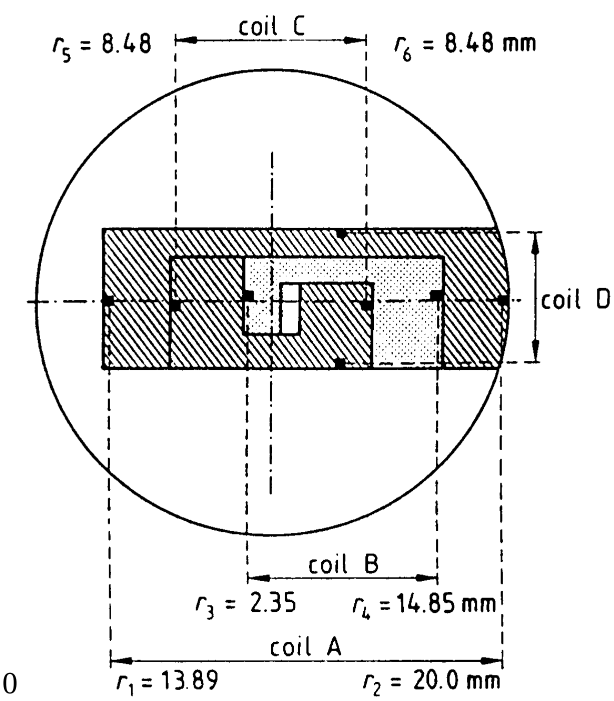
Coil B: Dipole Bucking Coil

Coil C: For compensating any angular misalignment of Coils A and B.

Condition for Bucking the Dipole field component:

$$N_A(r_2 - r_1) = N_B(r_3 + r_4)$$

HERA Quadrupole Coil



The compensated signal is:

$$V_{\text{bucked}} = V_A - V_B - V_c$$

Coil D: For compensating any angular misalignment of Coils.

Condition for Bucking the Dipole field component:

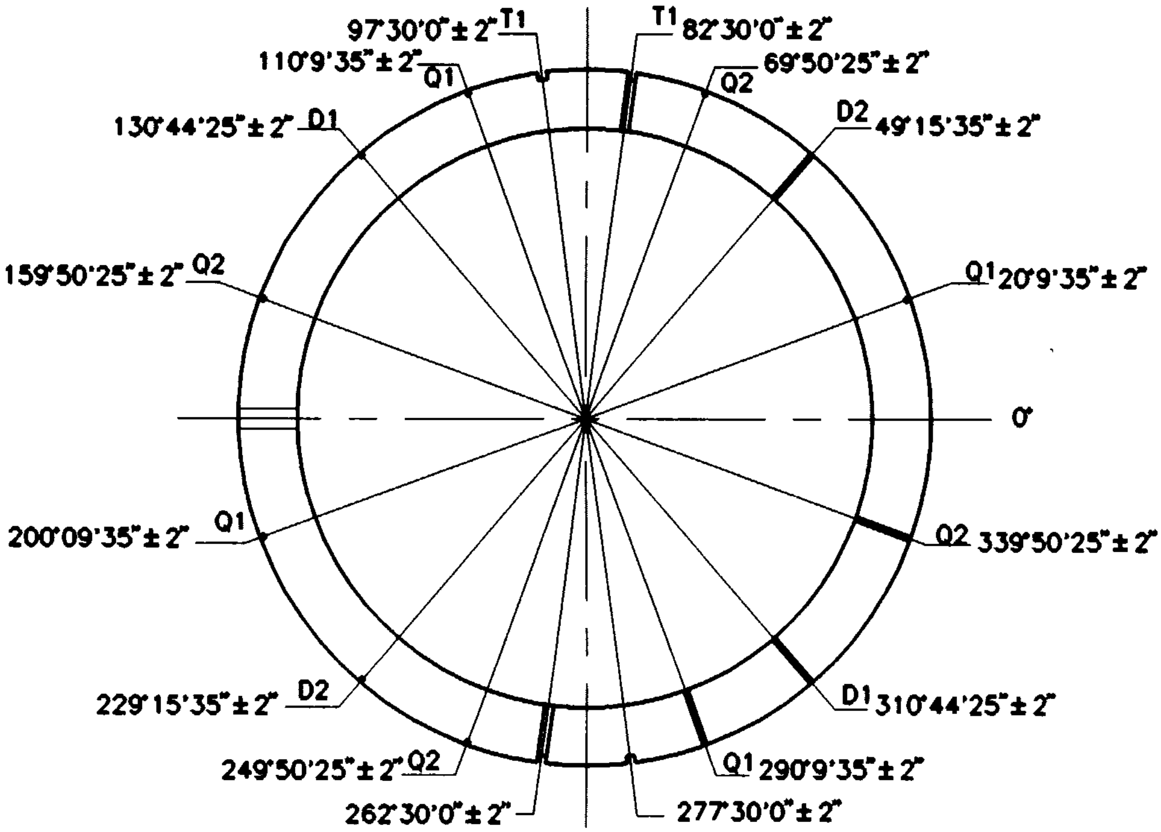
$$N_A(r_2 + r_1) - N_B(r_3 + r_4) - N_C(r_5 + r_6) = 0$$

Condition for Bucking the Quadrupole field component:

$$N_A(r_2^2 - r_1^2) - N_B(r_4^2 - r_3^2) - N_C(r_6^2 - r_5^2) = 0$$

Other designs also exist with a similar philosophy.

Example of a Practical Tangential Coil: the RHIC Coils



Tangential Winding: 15 degrees opening angle, 30 Turns
Dipole Buck Windings: 3 Turns each, at ±49.260 deg. wrt tangential
Quad Buck Windings: 3 Turns each, at ±24.840 deg. wrt tangential

Analysis is based on the Voltage signals, rather than the integrated voltage signals (the flux). The bucked signal is defined as:

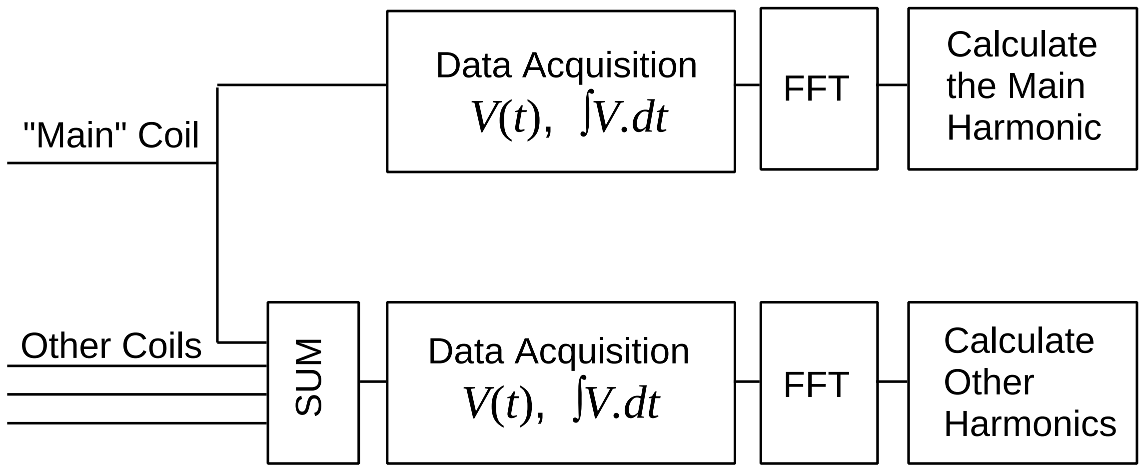
$$V_{\text{bucked}} = V_{\text{tangential}} + f_1 V_{\text{DB1}} + f_2 V_{\text{DB2}} + f_4 V_{\text{QB1}} + f_5 V_{\text{QB2}}$$

With the above design, $f_1 = f_2 = f_4 = f_5 = -1$ to buck out the dipole and the quadrupole components in the bucked signal.

All RHIC measuring coils are built with the same basic design. The radii of the windings are scaled to suit the aperture of the magnet to be measured.

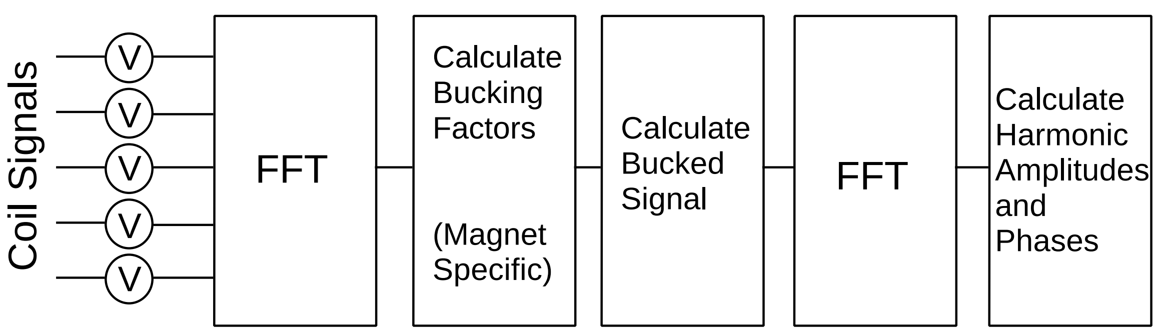
Analog Bucking

In Analog Bucking, the various coil signals are added before doing any analysis. The FFT is also carried out on one (or more) signal directly to get the main harmonic component. The summing circuit must be tuned to precisely cancel (a bucking factor of at least several hundred) the dipole and the quadrupole components.



Digital Bucking

In digital bucking, all the coil signals are acquired without any summing. The summing coefficient for each signal is determined from a FFT analysis. These coefficients are then calculated based on which two harmonics are to be eliminated in the bucked signal. The bucked signal is then digitally constructed and Fourier analyzed to get the harmonics.



Bucking Algorithm for the RHIC Coil inVarious Magnets

Five Windings:

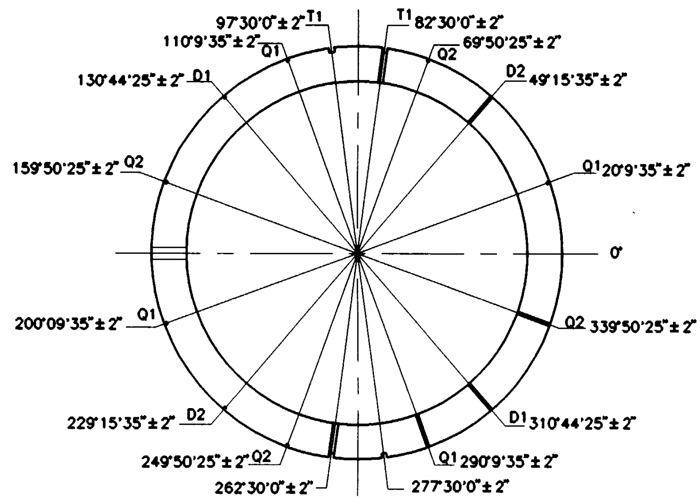
DB1: Dipole Coil

DB2: Dipole Coil

T: Tangential

QB1: Quad Coil

QB2: Quad Coil



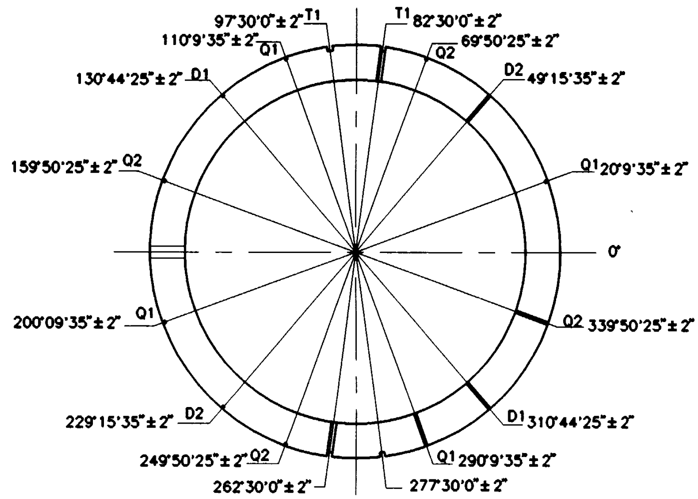
DB1,DB2: Sensitive to Dipole, Sextupole, Decapole, etc. terms.
QB1,QB2: Sensitive to Quadrupole, Dodecapole, etc. terms.
T: Sensitive to all harmonics of interest.

Goal: To buck the most dominant, and the next lower order harmonic for any magnet. (Not achieved for Octupole and Decapole magnets)

Magnet Type	**Use DB1,DB2 to buck**	**Use QB1,QB2 to buck**	**Calculation of Harmonics**
Dipole	Dipole	Quadrupole (optional)	♦ Dipole from DB1,DB2 ♦ Quad from QB1,QB2 (if used) ♦ Rest from Bucked signal.
Quadrupole	Dipole	Quadrupole	♦ Dipole from DB1,DB2 ♦ Quad from QB1,QB2 ♦ Rest from Bucked signal.
Sextupole	Sextupole	Quadrupole	♦ Sextupole from Tangential ♦ Quad from QB1,QB2 ♦ Rest from Bucked signal.
Decapole	Decapole	Quadrupole (Optional)	♦ Decapole from Tangential ♦ Quad from QB1,QB2 ♦ Rest from Bucked signal.
Dodecapole	Decapole	Dodecapole	♦ Dodecapole from Tangential ♦ Decapole from DB1,DB2 ♦ Rest from Bucked signal.

Calculation of the Bucked Signal

DB1: N_1, R_1, δ_1
DB2: N_2, R_2, δ_2
T: N_3, R_3, δ_3, Δ
QB1: N_4, R_4, δ_4
QB2: N_5, R_5, δ_5



$$
\begin{aligned}
V_{\text{bucked}} = V_{\text{tangential}} &+ f_1 V_{\text{DB1}} \\
&+ f_2 V_{\text{DB2}} + f_4 V_{\text{QB1}} \\
&+ f_5 V_{\text{QB2}}
\end{aligned}
$$

The values of the coefficients f_1, f_2, f_4, f_5 are calculated from the Fourier analysis of the individual signals in such a way as to completely cancel two of the harmonics chosen according to the type of the magnet being measured.

If n_1 is the harmonic to be cancelled with the DB1 and DB2 windings, then the design values of f_1 and f_2 are given by:

$$f_1 = \left(\frac{N_3}{N_1}\right)\left(\frac{R_3}{R_1}\right)^{n_1} \frac{\sin\{n_1(\delta_3 - \delta_2)\}}{\sin\{n_1(\delta_2 - \delta_1)\}} \sin\left(\frac{n_1 \Delta}{2}\right) \sin\left(\frac{n_1 \pi}{2}\right)$$

$$f_2 = \left(\frac{N_3}{N_2}\right)\left(\frac{R_3}{R_2}\right)^{n_1} \frac{\sin\{n_1(\delta_1 - \delta_3)\}}{\sin\{n_1(\delta_2 - \delta_1)\}} \sin\left(\frac{n_1 \Delta}{2}\right) \sin\left(\frac{n_1 \pi}{2}\right)$$

Similarly, if n_2 is the harmonic to be cancelled using the QB1 and QB2:

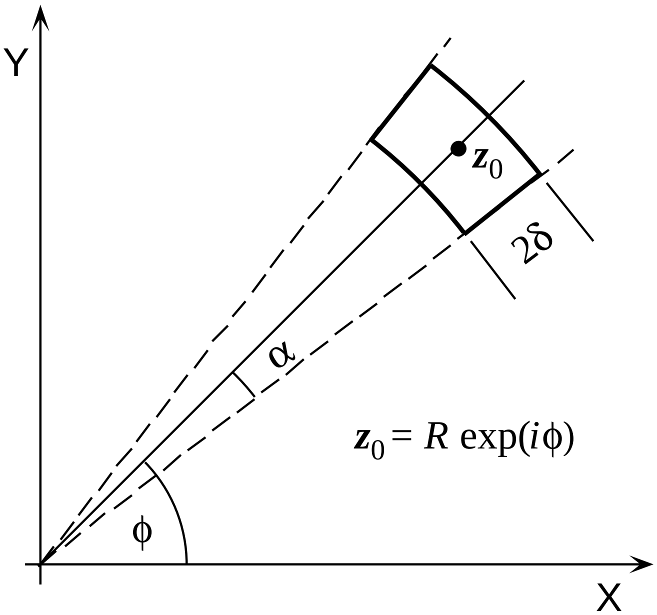
$$f_4 = \left(\frac{N_3}{2N_4}\right)\left(\frac{R_3}{R_4}\right)^{n_2} \frac{\sin\{n_2(\delta_3 - \delta_5)\}}{\sin\{n_2(\delta_5 - \delta_4)\}} \sin\left(\frac{n_2 \Delta}{2}\right) \sin\left(\frac{n_2 \pi}{4}\right)$$

$$f_5 = \left(\frac{N_3}{2N_5}\right)\left(\frac{R_3}{R_5}\right)^{n_2} \frac{\sin\{n_2(\delta_4 - \delta_3)\}}{\sin\{n_2(\delta_5 - \delta_4)\}} \sin\left(\frac{n_2 \Delta}{2}\right) \sin\left(\frac{n_2 \pi}{4}\right)$$

n_1	f_1	f_2	n_2	f_4	f_5
1	–1.00	–1.00	2	–1.00	–1.00
3	–2.26	–2.26	6	–2.06	–2.06
5	+7.57	+7.57	—	—	—

Effect of the Finite Size of the Coil Windings

In practice, the coil windings are not point-like. To accommodate the necessary number of turns, the winding must have a finite cross section. This could introduce errors in the measurement of the amplitude of the harmonics. Although typical winding cross sections are rectangular, it is convenient to approximate it with a sector of an annulus, as shown in figure. The winding is assumed to have an angular width of (2α) and thickness (2δ). The mean position of the winding is denoted by $\boldsymbol{z}_0 = R.\exp(i\phi)$. The sensitivity factor of the winding to the $2n$-pole term involves the quantity $\boldsymbol{z}^n$. The value of $\boldsymbol{z}^n$ averaged over the cross section of the winding is:



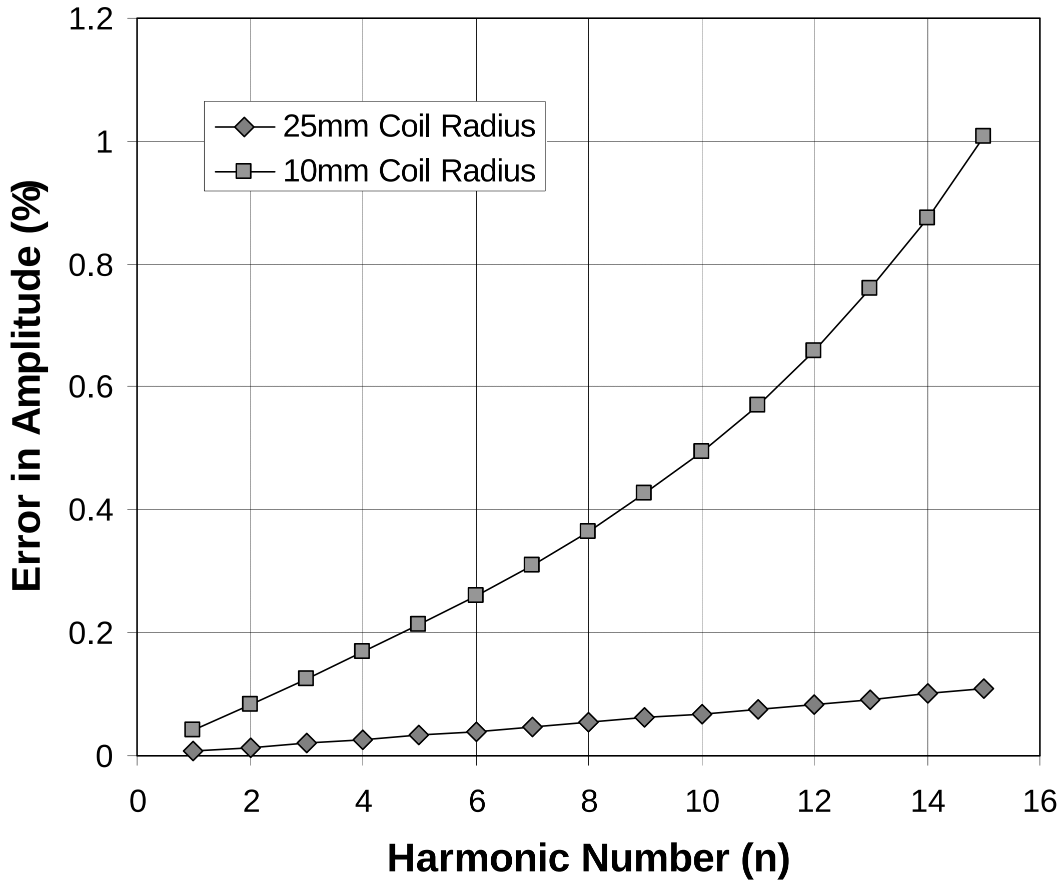
$$(\boldsymbol{z}^n)_{avg.} = \frac{\int_{R-\delta}^{R+\delta} r^n dr \int_{\phi-\alpha}^{\phi+\alpha} \exp(in\phi)\, d\phi}{(2\alpha)(2\delta)} = \frac{\left[(R+\delta)^{n+1} - (R-\delta)^{n+1}\right]\left[e^{in(\phi+\alpha)} - e^{in(\phi-\alpha)}\right]}{4in(n+1)\alpha\delta}$$

$$= R^n \exp(in\phi) \frac{\sin(n\alpha)}{n\alpha} \frac{\left[\left(1+\frac{\delta}{R}\right)^{n+1} - \left(1-\frac{\delta}{R}\right)^{n+1}\right]}{2(n+1)(\delta / R)}$$

$$= \boldsymbol{z}_0^n \frac{\sin(n\alpha)}{n\alpha} \frac{\left[\left(1+\frac{\delta}{R}\right)^{n+1} - \left(1-\frac{\delta}{R}\right)^{n+1}\right]}{2(n+1)(\delta / R)}$$

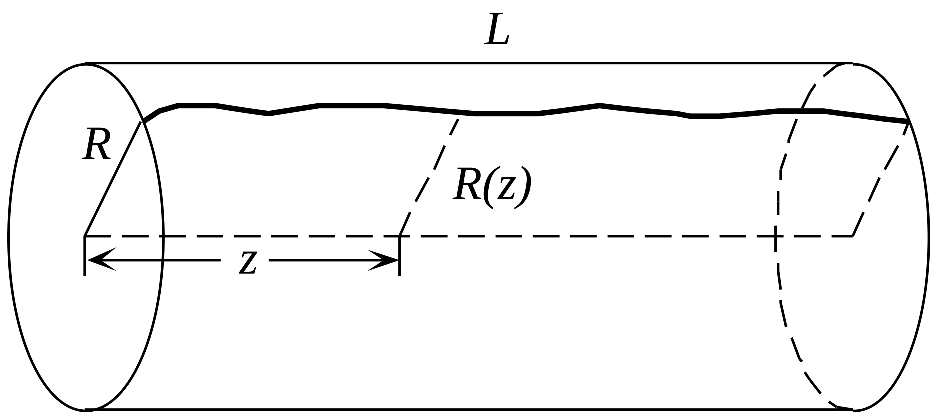
If the winding is assumed to be point like and located at the geometric center, the above formula gives the error in estimating the amplitude of the $2n$-pole term. Expanding in a power series, it can be shown that the leading correction terms are of the second order in α and (δ/R).

Effect of a 1 mm ´ 1 mm Winding Cross Section



For a 25 mm average radius of the measuring coil, the errors introduced with a 1 mm wide and 1 mm thick winding are negligible for all the harmonics of interest. For a smaller radius coil, the errors are more pronounced, as expected. However, even for a 10 mm radius coil, the errors for the harmonics of interest are still less than one percent. The finite size may, however, be a serious limitation in measuring the transfer function of higher multipole magnets (such as sextupoles, octupoles, etc. correctors) with a small diameter measuring coil.

Random Variation of Winding Radius Along the Length



Let us consider one segment of the coil loop in either a radial or a tangential coil. The radius is assumed to vary randomly along the length L of the coil with a mean value of R_c and a standard deviation of σ_R. The effective sensitivity factor of the coil for the n-th order harmonic is proportional to R^n.

$$R(z) = R_c + \varepsilon(z); \quad \frac{1}{L}\int_0^L R(z)\,dz = R_c; \quad \int_0^L \varepsilon(z)\,dz = 0; \quad \sigma_R^2 = \frac{1}{L}\int_0^L \left[R(z) - R_c\right]^2 dz$$

$$\frac{1}{L}\int_0^L \left[R(z)\right]^n dz = \frac{R_c^n}{L}\int_0^L \left[1 + \frac{n}{R_c}\varepsilon(z) + \frac{n(n-1)}{2R_c^2}\varepsilon^2(z) + \cdots\right] dz$$

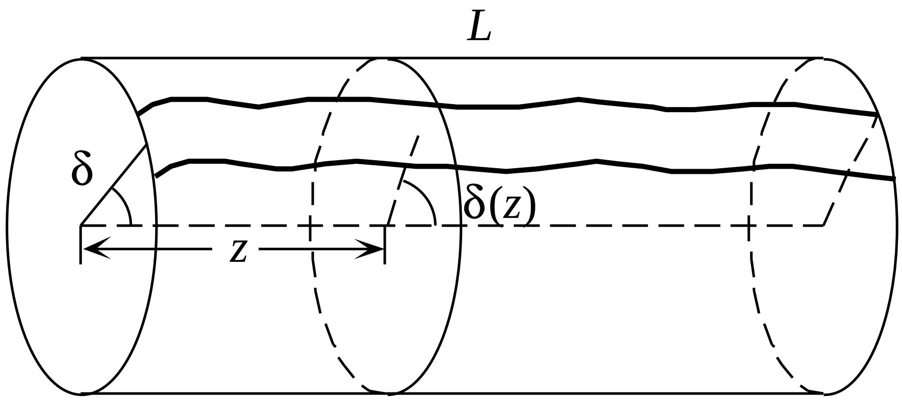
$$\approx R_c^n\left[1 + \frac{n(n-1)}{2}\left(\frac{\sigma_R}{R_c}\right)^2\right]$$

The sensitivity factor for the n-th harmonic is given by:

$$\boldsymbol{K}_n \approx \boldsymbol{K}_n^{ideal}\left[1 + \frac{n(n-1)}{2}\left(\frac{\sigma_R}{R_c}\right)^2\right]$$

In order to keep the error in the amplitude of the n = 15 term less than 1%, we should have $\sigma_R \leq 10^{-2} R_c$. A somewhat tighter tolerance may be required if such a coil is to be used to determine the transfer function in a magnet of higher multipolarity, such as in a dodecapole corrector magnet.

Random Variation of Angular Position Along the Length (Twist)



Let us consider a tangential coil in which the radii and opening angle are uniform along the length. However, the angular position, δ, is assumed to vary randomly along the length L of the coil with a mean value of δ_c and a standard deviation of σ_δ.

$$\delta(z) = \delta_c + \varepsilon(z); \quad \frac{1}{L}\int_0^L \delta(z)\,dz = \delta_c; \quad \int_0^L \varepsilon(z)\,dz = 0; \quad \sigma_\delta^2 = \frac{1}{L}\int_0^L [\delta(z) - \delta_c]^2\,dz$$

The flux seen by the coil for the $2n$-pole field is:

$$\Phi_n(t) \propto \frac{1}{L}\int_0^L C(n)\sin(n\omega t + n\delta_c + n\varepsilon(z) - n\alpha_n)\,dz$$

Expanding $\sin[n\varepsilon(z)]$ and $\cos[n\varepsilon(z)]$ in power series and retaining only the terms up to the second order, it is easy to show that:

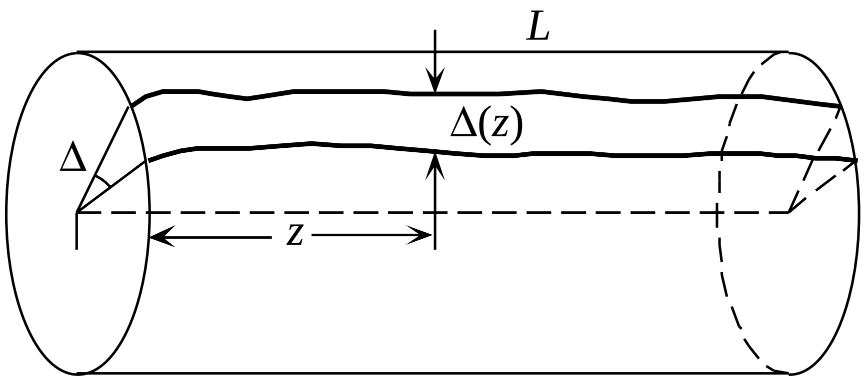
$$\Phi_n(t) \propto \frac{1}{L}C(n)\sin(n\omega t + n\delta_c - n\alpha_n)\int_0^L \left[1 - \frac{n^2}{2}\varepsilon^2(z)\right]dz + \frac{n}{L}C(n)\cos(n\omega t + n\delta_c - n\alpha_n)\int_0^L \varepsilon(z)\,dz$$

$$= C(n)\sin(n\omega t + n\delta_c - n\alpha_n)\left[1 - \frac{n^2}{2}\sigma_\delta^2\right]$$

The sensitivity factor for the n-th harmonic is given by:

$$\boldsymbol{K}_n \approx \boldsymbol{K}_n^{ideal}\left[1 - \frac{n^2}{2}\sigma_\delta^2\right]$$

Random Variation in Opening Angle Along the Length (Tangential Coil)



Let us consider a tangential coil in which the radii and the mean angular position are uniform along the length. However, the opening angle, Δ, is assumed to vary randomly along the length L of the coil with a mean value of Δ_c and a standard deviation of σ_Δ.

$$\Delta(z) = \Delta_c + \varepsilon(z); \quad \frac{1}{L}\int_0^L \Delta(z)\,dz = \Delta_c; \quad \int_0^L \varepsilon(z)\,dz = 0; \quad \sigma_\Delta^2 = \frac{1}{L}\int_0^L \left[\Delta(z) - \Delta_c\right]^2 dz$$

The flux seen by the coil for the 2n-pole field is:

$$\Phi_n(\theta) \propto \frac{1}{L}\int_0^L \sin\left(\frac{n\Delta(z)}{2}\right) dz$$

Expanding sin[$n\varepsilon(z)/2$] and cos[$n\varepsilon(z)/2$] in power series and retaining only up to the second order terms, it is easy to show that:

$$\Phi_n(\theta) \propto \frac{1}{L}\sin\left(\frac{n\Delta_c}{2}\right)\int_0^L \left[1 - \frac{n^2}{8}\varepsilon^2(z)\right] dz + \frac{n}{2L}\cos\left(\frac{n\Delta_c}{2}\right)\int_0^L \varepsilon(z)\,dz$$

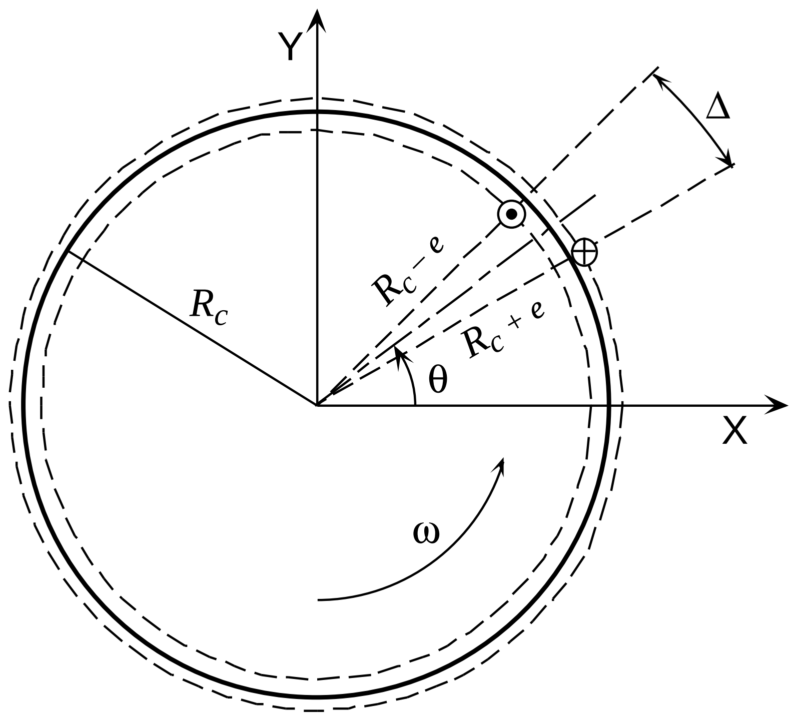
$$\Phi_n(\theta) \propto \sin\left(\frac{n\Delta_c}{2}\right)\left[1 - \frac{n^2}{8}\sigma_\Delta^2\right]$$

The sensitivity factor for the n-th harmonic is given by:

$$\boldsymbol{K}_n \approx \boldsymbol{K}_n^{ideal}\left[1 - \frac{n^2}{8}\sigma_\Delta^2\right]$$

Imperfect Tangential Coil : Unequal Radii of the Two Grooves

Let us consider a slightly imperfect tangential coil where the two sides of the coil loop are not at the same radius. Such an imperfection can be real, resulting from unequal depths of grooves in the coil form. Even with a perfectly built coil, such an imperfection will be apparent if the rotation axis does not exactly coincide with the geometric center of the windings. It is assumed here that the wires are located at radii of $R_c - \varepsilon$ and $R_c + \varepsilon$.



$$\Phi(\theta) = \mathrm{Re}\left[\sum_{n=1}^{\infty} \boldsymbol{K}_n \exp(in\theta) C(n) \exp(-in\alpha_n)\right];$$

$$\boldsymbol{K}_n = \left(\frac{NLR_{ref}}{n}\right)\left\{\left(\frac{\boldsymbol{z}_{2,0}}{R_{ref}}\right)^n - \left(\frac{\boldsymbol{z}_{1,0}}{R_{ref}}\right)^n\right\}$$

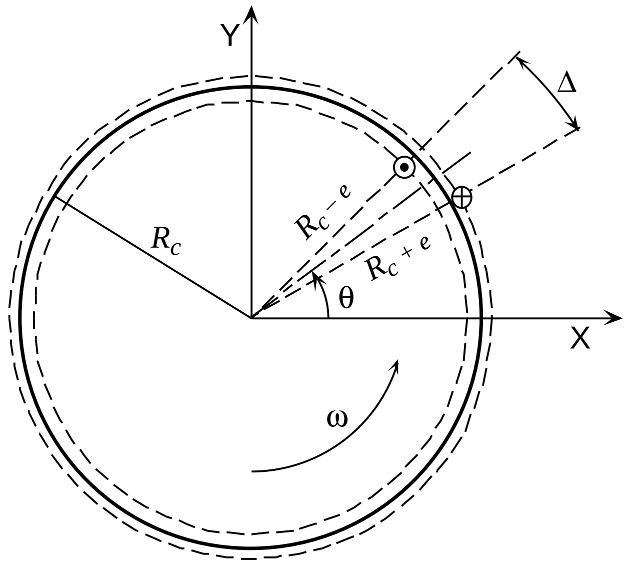
$$\boldsymbol{z}_{1,0} = (R_c - \varepsilon)\exp(i\Delta/2); \quad \boldsymbol{z}_{2,0} = (R_c + \varepsilon)\exp(-i\Delta/2)$$

$$\boldsymbol{z}_{2,0}^n - \boldsymbol{z}_{1,0}^n = (R_c + \varepsilon)^n \exp(-in\Delta/2) - (R_c - \varepsilon)^n \exp(in\Delta/2)$$

$$\approx 2R_c^n\left[-i\sin\left(\frac{n\Delta}{2}\right) + \left(\frac{n\varepsilon}{R_c}\right)\cos\left(\frac{n\Delta}{2}\right)\right]$$

The first term can be identified to be related to the sensitivity of the perfect coil. The second term implies that both amplitude and phase errors are introduced in the sensitivity factor. Also, for coils such as the dipole coil with $\Delta = \pi$, the flux for a perfect coil is zero for even harmonics. This is no longer the case with an imperfect coil. However, the allowed terms for a dipole coil are not affected since $\cos(n\pi/2) = 0$ for odd values of n.

Imperfect Tangential Coil : Unequal Radii of the Two Grooves



$$\mathbf{z}_{2,0}^{n} - \mathbf{z}_{1,0}^{n} = (R_c + \varepsilon)^n e^{-in\Delta/2} - (R_c - \varepsilon)^n e^{in\Delta/2} \approx 2R_c^n \left[-i\sin\left(\frac{n\Delta}{2}\right) + \left(\frac{n\varepsilon}{R_c}\right)\cos\left(\frac{n\Delta}{2}\right) \right]$$

Assuming that $\sin(n\Delta/2)$ is not zero, as is the case for the harmonics of interest in a practical tangential coil ($\Delta \approx 15°$):

$$\mathbf{z}_{2,0}^{n} - \mathbf{z}_{1,0}^{n} \approx -2iR_c^n \sin\left(\frac{n\Delta}{2}\right)\left[1 + i\left(\frac{n\varepsilon}{R_c}\right)\cot\left(\frac{n\Delta}{2}\right)\right] = -2iR_c^n \sin\left(\frac{n\Delta}{2}\right) A_n \exp(in\lambda_n)$$

where A_n is an amplitude correction term and λ_n is a phase error given by:

$$A_n = \sqrt{1 + \left(\frac{n\varepsilon}{R_c}\right)^2 \cot^2\left(\frac{n\Delta}{2}\right)} \approx 1 + \frac{n^2}{2}\left(\frac{\varepsilon}{R_c}\right)^2 \cot^2\left(\frac{n\Delta}{2}\right)$$

$$\lambda_n = \left(\frac{1}{n}\right)\tan^{-1}\left[\left(\frac{n\varepsilon}{R_c}\right)\cot\left(\frac{n\Delta}{2}\right)\right] \approx \left(\frac{\varepsilon}{R_c}\right)\cot\left(\frac{n\Delta}{2}\right)$$

The amplitude error is of the second order in (ε/R_c) and can generally be neglected. For typical values of (ε/R_c) ~ 10^{-3}, the phase error could be several milli-radians for the lowest order harmonics. The phase error reduces with the order of the harmonic as roughly ($1/n$). The expression for flux is given by:

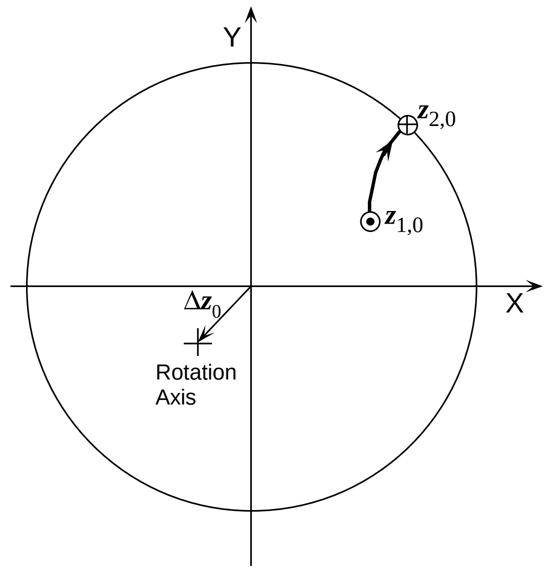
$$\Phi(\theta) \approx \sum_{n=1}^{\infty} \frac{2NLR_{ref}}{n}\left(\frac{R_c}{R_{ref}}\right)^n \sin\left(\frac{n\Delta}{2}\right) A_n C(n) \sin\{n\theta - n(\alpha_n - \lambda_n)\}$$

Offset in the Rotation Axis

$$\boldsymbol{K}_n \propto \boldsymbol{z}_{2,0}^n - \boldsymbol{z}_{1,0}^n$$

$$\boldsymbol{K}_n' \propto (\boldsymbol{z}_{2,0} + \Delta\boldsymbol{z}_0)^n - (\boldsymbol{z}_{1,0} + \Delta\boldsymbol{z}_0)^n$$

$$\frac{\Delta\boldsymbol{K}_n}{\boldsymbol{K}_n} = \frac{(\boldsymbol{z}_{2,0} + \Delta\boldsymbol{z}_0)^n - (\boldsymbol{z}_{1,0} + \Delta\boldsymbol{z}_0)^n}{\boldsymbol{z}_{2,0}^n - \boldsymbol{z}_{1,0}^n} - 1$$



The sensitivity factor for the dipole term is not affected. For other harmonics,

$$\left(\frac{\Delta\boldsymbol{K}_n}{\boldsymbol{K}_n}\right)_{\text{tangential}} = \sum_{k=1}^{n-1}\left[\frac{n!}{k!(n-k)!}\right]\left(\frac{\Delta\boldsymbol{z}_0}{R_c}\right)^k \frac{\sin\left(\frac{(n-k)\Delta}{2}\right)}{\sin\left(\frac{n\Delta}{2}\right)}$$

$$\left(\frac{\Delta\boldsymbol{K}_n}{\boldsymbol{K}_n}\right)_{\text{radial}} = \sum_{k=1}^{n-1}\left[\frac{n!}{k!(n-k)!}\right](\Delta\boldsymbol{z}_0)^k \frac{R_2^{n-k} - R_1^{n-k}}{R_2^n - R_1^n}$$

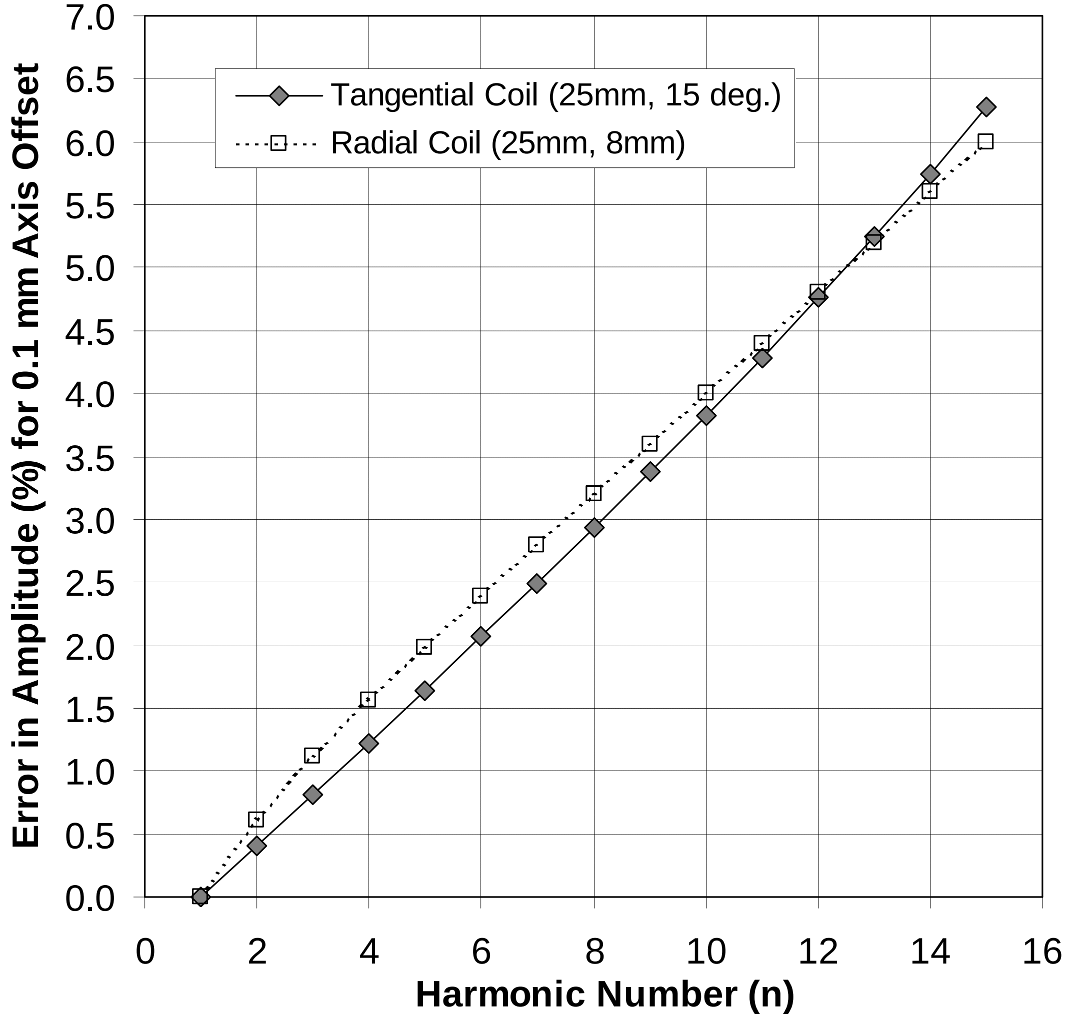
Usually, a first order approximation is adequate:

$$\frac{\Delta\boldsymbol{K}_n}{\boldsymbol{K}_n} \approx n(\Delta\boldsymbol{z}_0)\left[\frac{\boldsymbol{z}_{2,0}^{n-1} - \boldsymbol{z}_{1,0}^{n-1}}{\boldsymbol{z}_{2,0}^n - \boldsymbol{z}_{1,0}^n}\right]; \qquad \left(\frac{\Delta\boldsymbol{K}_n}{\boldsymbol{K}_n}\right)_{\text{tangential}} \approx n\left(\frac{\Delta\boldsymbol{z}_0}{R_c}\right)\frac{\sin\left[\frac{(n-1)\Delta}{2}\right]}{\sin\left[\frac{n\Delta}{2}\right]}$$

$$\left(\frac{\Delta\boldsymbol{K}_n}{\boldsymbol{K}_n}\right)_{\text{radial}} \approx n(\Delta\boldsymbol{z}_0)\left[\frac{R_2^{n-1} - R_1^{n-1}}{R_2^n - R_1^n}\right]$$

These results may also be used to estimate the effect of a "bow" or a bend in the measuring coil. Different sections of such a coil will rotate about an axis which is offset from the geometric center by different amounts. An upper bound on the resulting effect can be obtained by equating $\Delta\boldsymbol{z}_0$ to the total bend in the measuring coil.

Offset in the Rotation Axis



Percent error in the amplitudes of various harmonics with a 0.1 mm offset in the rotation axis. The tangential coil is assumed to have a radius of 25 mm and opening angle of 15 degrees, while the radial coil is assumed to have the radii R_2 = 25 mm and R_1 = 8 mm (~ R_1/3).

Systematic Errors in Coil Parameters (Calibration Errors)

The coil parameters of primary interest are the radius (R), the angular position at the start of the data acquision (δ), and in the case of a tangential coil, the opening angle (Δ). A systematic error in the knowledge of these parameters will result in systematic errors in the calculation of the field parameters, namely the amplitudes $C(n)$ and the phase angles α_n.

Systematic Error in the Radius:

$$\boldsymbol{K}_n \propto R^n; \quad \therefore \frac{\Delta \boldsymbol{K}_n}{\boldsymbol{K}_n} = n\left(\frac{\Delta R}{R}\right)$$

where $\boldsymbol{K}_n$ is the sensitivity factor for the n-th harmonic and ΔR is the systematic error in the radius R. For a $(\Delta R/R) \sim 10^{-3}$, the systematic error in the amplitude of the 20-pole term will be ~ 1%.

Systematic Error in the angular Position:

A systematic error, ε_δ, in the initial angular position, δ, leads to the same error in the determination of all the phase angles. This would give rise to skew terms in a purely normal magnet, and vice versa.

$$\alpha_n \to \alpha_n - \varepsilon_\delta; \quad C(n)\exp(-in\alpha_n) \to C(n)\exp(-in\alpha_n)\left[\cos(n\varepsilon_\delta) + i\sin(n\varepsilon_\delta)\right]$$

The multipoles in a magnet are generally expressed in a reference frame where the main field component has a zero phase angle. In this case, there will be no systematic error in the multipoles, since the phase angles relative to the main field still remain the same. However, when accurate determination of the field direction of the main component is required, such a systematic error is unacceptable. Efforts must be made to periodically check the calibration, and/or correct for the errors by making measurements from the lead and non-lead ends of the magnet. For a $2m$-pole magnet, the measured phase angles from the lead and the non-lead ends are:

$$\alpha_{\text{lead}} = \alpha_m - \varepsilon_\delta; \quad \alpha_{\text{non-lead}} = \left[1 + (-1)^m\right]\left(\frac{\pi}{2m}\right) - \alpha_m - \varepsilon_\delta$$

$$\therefore \varepsilon_\delta = \left(\tfrac{1}{2}\right)\left[\{1 + (-1)^m\}\left(\tfrac{\pi}{2m}\right) - \alpha_{\text{lead}} - \alpha_{\text{non-lead}}\right]$$

$$\alpha_m = \left(\tfrac{1}{2}\right)\left[\alpha_{\text{lead}} - \alpha_{\text{non-lead}} + \{1 + (-1)^m\}\left(\tfrac{\pi}{2m}\right)\right]$$

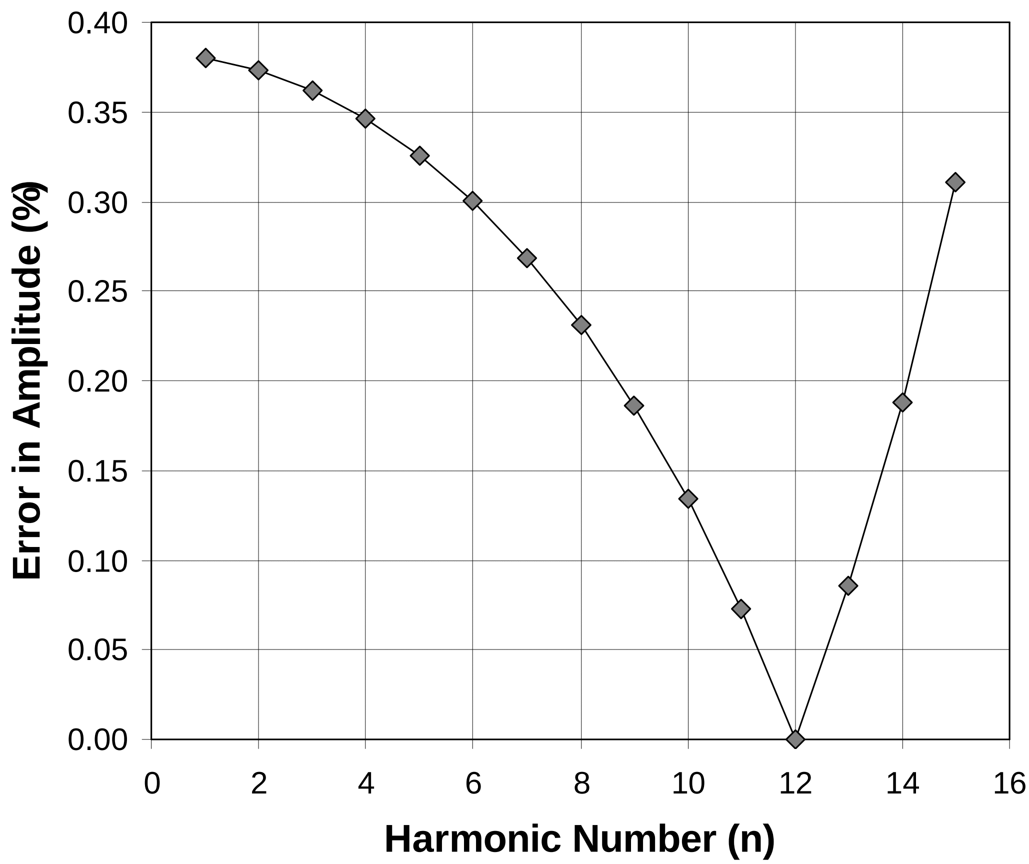
Systematic Error in the Opening Angle of a Tangential Coil

The sensitivity factor, $\boldsymbol{K}_n$, of a tangential coil depends on the opening angle, Δ, as:

$$\boldsymbol{K}_n \propto \sin\left(\frac{n\Delta}{2}\right)$$

For a systematic error ε_Δ in the opening angle,

$$\frac{\Delta \boldsymbol{K}_n}{\boldsymbol{K}_n} = \left(\frac{n}{2}\right)\cot\left(\frac{n\Delta}{2}\right)\varepsilon_\Delta$$



Effect of a 1 mrad systematic error in the determination of the opening angle of a tangential coil (Δ = 15 degrees). The error is significant for the lower order harmonics. In a typical 5 winding tangential coil system, the dipole and the quadrupole terms are obtained from the dipole (Δ = 180 degrees) and the quadrupole windings (Δ = 90 degrees) which are practically insensitive to this error.

Effect of a Finite Averaging Time

In the acquisition of voltage data from the RHIC tangential coils, the signals are averaged over one power line cycle to get rid of any AC noise on the signals. At a typical angular speed of one revolution every 3.5 seconds and a power line frequency of 60Hz, the coil rotates about 1.7 degrees during one power line cycle. This motion during data integration can cause errors.

If Δt is the averaging time, the n-th harmonic component in the measured voltage signal is:

$$
\begin{aligned}
V_n(t) &\propto \frac{1}{\Delta t}\int_t^{t+\Delta t} \cos(n\omega t + n\delta - n\alpha_n)\,dt \\
&= \frac{1}{\Delta t}\left[\sin(n\omega t + n\delta - n\alpha_n + n\omega\Delta t) - \sin(n\omega t + n\delta - n\alpha_n)\right] \\
&= \frac{\sin(n\omega\Delta t/2)}{n\omega\Delta t/2}\cos\left[n\omega t + n\left(\delta + \frac{\omega\Delta t}{2}\right) - n\alpha_n\right]
\end{aligned}
$$

Amplitude Correction Factor = $\dfrac{\sin(n\omega\Delta t/2)}{n\omega\Delta t/2} \approx 1 - 1.64 n^2\left(\dfrac{\Delta t}{T}\right)^2$

Effective Angle Calibration = $\delta' = \delta + (\omega\Delta t/2) = \delta + \pi\left(\dfrac{\Delta t}{T}\right)$

where T is the period of rotation. For Δt = 1/60 sec. and T = 3.5 sec, $(\Delta t/T)$ ~ 4.8×10–3.

The amplitude error is 0.004% for the dipole term and is 0.84% for the 30-pole term. This effect is negligible. However, the angle calibration is affected by roughly 0.86 degrees (15 mrad). Fortunately, the error is harmonic independent, and can be absorbed in the calibration of the coil, as long as the angular velocity is kept the same. Considerable error will result, for example, if the coil were to rotate in the opposite direction ($\omega \rightarrow -\omega$).

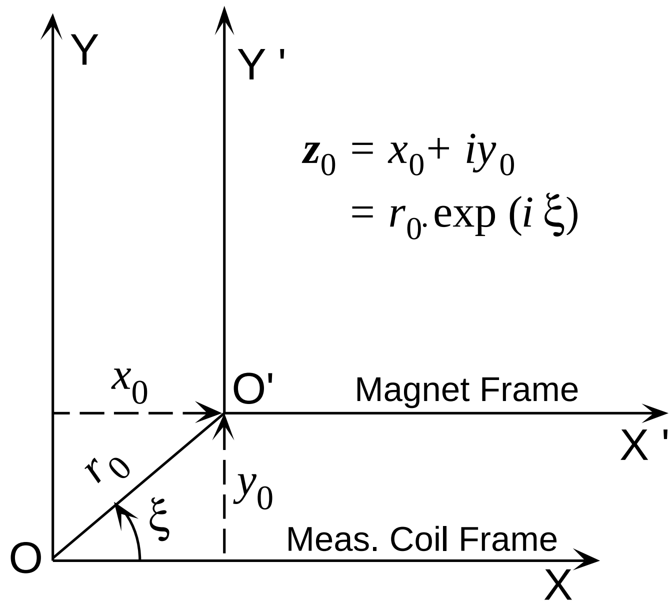
In practice, the coil rotation period may not be the same during calibration and measurements. It is necessary, therefore, to specify the rotation speed of the measuring coil along with the angular parameters. Corrections must be applied to the calibration values based on the actual rotation speed during the measurements according to the above equations.

Rotation Axis Different from the Magnetic Axis

If the rotation axis of the coil is not coincident with the magnetic axis of the magnet, the measured harmonic coefficients are affected by feed down.

O: Center of Measuring Coil
O′: Magnetic Center

$\boldsymbol{z}_0 = x_0 + iy_0$ = Location of magnetic center in the measuring coil frame.



If $C(n)$ and α_n are the measured parameters in the measuring coil frame, then the parameters in the magnet's frame are given by:

$$C'(n)\exp(-in\alpha'_n) = \sum_{k=n}^{\infty} C(k)\exp(-ik\alpha_k)\frac{(k-1)!}{(n-1)!(k-n)!}\left(\frac{\boldsymbol{z}_0}{R_{ref}}\right)^{k-n}$$

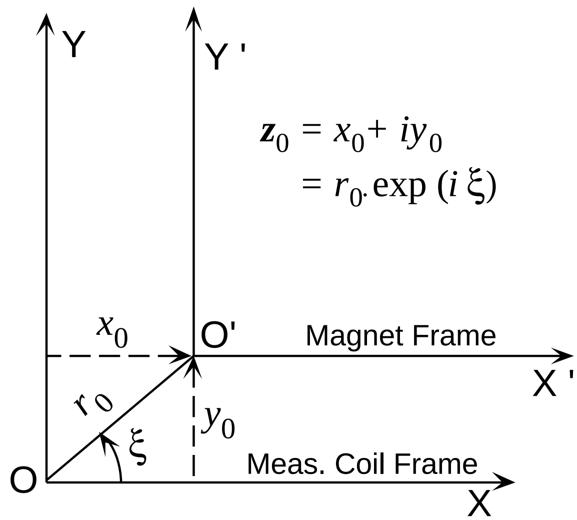
Magnetic center is defined as the location where an appropriate harmonic is zero. For example, for a $2m$-pole magnet other than a dipole, the magnetic center is defined as the location where the $2(m\text{–}1)$-pole term is zero.

$$\begin{aligned} C'(m-1)e^{-i(m-1)\alpha'_{m-1}} &= C(m-1)e^{-i(m-1)\alpha_{m-1}} + (m-1)C(m)e^{-im\alpha_m}\left(\frac{\boldsymbol{z}_0}{R_{ref}}\right) \\ &\quad + \frac{m(m-1)}{2}C(m+1)e^{-i(m+1)\alpha_{m+1}}\left(\frac{\boldsymbol{z}_0}{R_{ref}}\right)^2 \\ &\quad + \frac{m(m^2-1)}{6}C(m+2)e^{-i(m+2)\alpha_{m+2}}\left(\frac{\boldsymbol{z}_0}{R_{ref}}\right)^3 + \cdots \\ &= 0 \end{aligned}$$

For most magnets, terms other than $C(m)$ are small. For small offsets,

$$\left(\frac{\boldsymbol{z}_0}{R_{ref}}\right) \approx -\frac{1}{(m-1)}\frac{C(m-1)\exp\{-i(m-1)\alpha_{m-1}\}}{C(m)\exp(-im\alpha_m)};\ m \neq 1;\ |\boldsymbol{z}_0| << R_{ref}$$

Rotation Axis Different from the Magnetic Axis

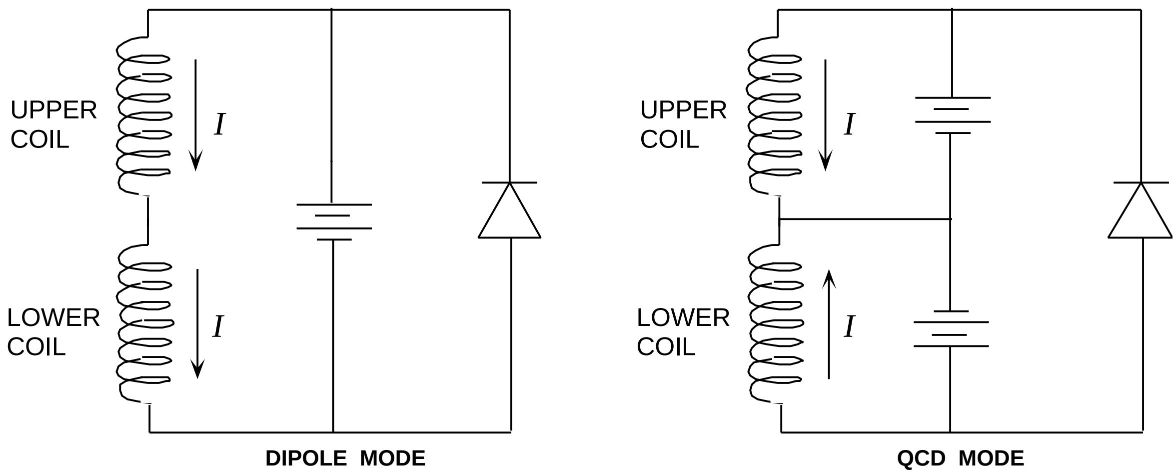


$$C'(m-1)e^{-i(m-1)\alpha'_{m-1}} = C(m-1)e^{-i(m-1)\alpha_{m-1}} + (m-1)C(m)e^{-im\alpha_m}\left(\frac{\boldsymbol{z}_0}{R_{ref}}\right)$$
$$+\frac{m(m-1)}{2}C(m+1)e^{-i(m+1)\alpha_{m+1}}\left(\frac{\boldsymbol{z}_0}{R_{ref}}\right)^2$$
$$+\frac{m(m^2-1)}{6}C(m+2)e^{-i(m+2)\alpha_{m+2}}\left(\frac{\boldsymbol{z}_0}{R_{ref}}\right)^3+\cdots$$

For dipole magnets, no “natural” definition of a center can be used. Various strategies are used to define the center of a dipole magnet. For example, one could argue that the very high order unallowed terms are not sensitive to small construction errors, and hence must be zero. If so, one could pick *m* in the above expression to be a sufficiently high order allowed term and calculate the center by requiring $C'(m–1)$ to be zero. Of course, this works only if $C(m)$ itself has sufficient strength. Because of the large values of *m*, the measured coefficients are of comparable strengths for both the allowed and the unallowed harmonics, even with relatively small offsets. As a result, it is often necessary to use higher order terms to calculate the offset. In many cases, ambiguous results may be obtained because of the non-linear nature of the equations. To resolve this, it is best to find a offset that will simultaneously minimize several unallowed harmonics, rather than just one.

Other strategies for dipoles include the hysteretic centering, or minimization of current dependence of quadrupole terms (cold data) or an “ugly quad” method, used with much success for both warm and cold data at RHIC.

The Quadrupole Configured Dipole ("Ugly Quad") Method

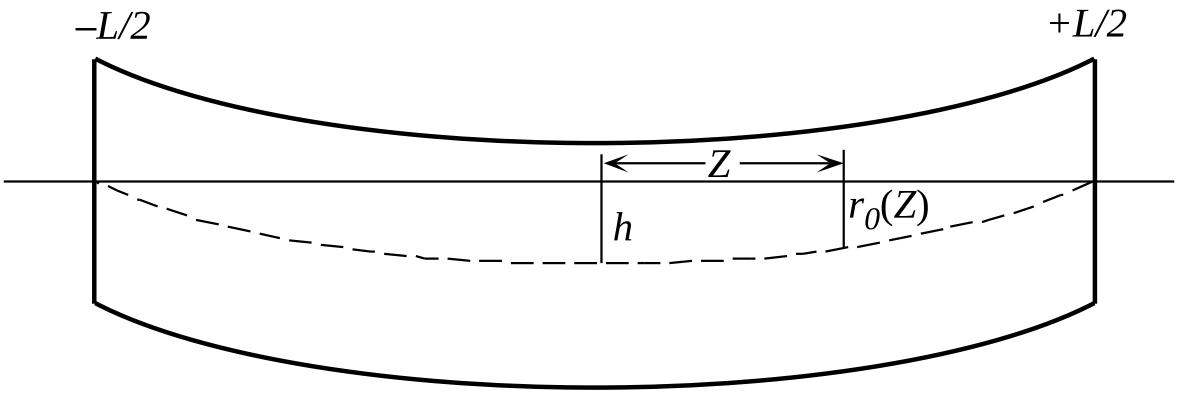


The quadrupole configured dipole method relies on powering the two coil halves of a dipole magnet with opposite currents to produce a strong skew quadrupole field, instead of a dipole field, as shown above. This requires a center tap connection on the magnet. The allowed harmonics are now the skew quadrupole, skew octupole, skew dodecapole, and so on. Several of these allowed harmonics are quite strong, and feed down from any one of them could be used to calculate the center.

Since two separate power supplies are required in this mode, it is important to balance the current in the two halves with great accuracy, otherwise a spurious dipole field will also be generated that would affect centering calculations. The sensitivity to any current mismatch can be greatly minimized by using feed down from the skew octupole term, rather than the dominant skew quadrupole term.

This method has been used at RHIC with great success. The results from QCD method have very little noise (typically only a few microns) and agree very well with the centers calculated by making high order unallowed terms zero.

Sag of the Measuring Coil Due to its Own Weight



For a long and thin measuring coil, the weight of the coil itself may be enough to cause a sagitta in the coil. In this case, each subsection of the coil rotates about its local geometric center. However, the location of this center varies along the length of the magnet, as shown by the dashed line in the above figure. This is different from a "bow" or bend in the coil. Various subsections of the coil see harmonics that are in a frame which is slightly displaced from the adjacent subsections. If $r_0(Z)$ is the vertically downward offset at axial position Z, the measured coefficients are given by:

$$C'(n)\exp(-in\alpha'_n) = \sum_{k=n}^{\infty} C(k)\exp(-ik\alpha_k)\frac{(k-1)!\exp\left\{-i(k-n)\frac{\pi}{2}\right\}}{(n-1)!(k-n)!}\left(\frac{1}{L}\right)\int_{-L/2}^{L/2}\left(\frac{r_0(Z)}{R_{ref}}\right)^{k-n} dZ$$

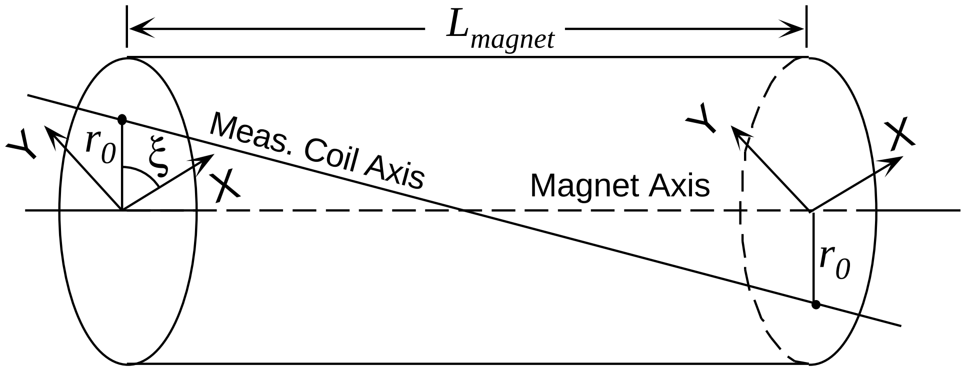
For a parabolic profile given by $r_0(Z) = h\left[1 - \frac{Z^2}{(L/2)^2}\right]$, it can be shown that:

$$C'(n)\exp(-in\alpha'_n) = \sum_{k=n}^{\infty} C(k)\exp(-ik\alpha_k)\frac{(k-1)!\exp\left\{-i(k-n)\frac{\pi}{2}\right\}}{(n-1)!(k-n)!}\frac{[2(k-n)]!!}{[2(k-n)+1]!!}\left(\frac{h}{R_{ref}}\right)^{k-n}$$

where $(2k)!! = 2.4.6.8. \ldots 2k$, $(2k+1)!! = 1.3.5. \ldots (2k+1)$ and $0!! = 1$.

For small values of h, the effect of sag is the same as a uniform displacement of the coil by an amount $(2h/3)$. If the measured data are corrected for the offset of the measuring coil axis, most of the errors due to sag are also subtracted out, except for terms of second and higher orders in (h/R_{ref}) which can be neglected for reasonable values of h.

Measuring Coil Axis Tilted wrt the Magnet Axis



Average displacement of the measuring coil = 0
Displacements at the two ends are given by (r_0, ξ) and (r_0, $\xi+\pi$)

If the field quality of the magnet is uniform along the length, then the odd orders of feed down from one half of the magnet will be cancelled by the corresponding feed down from the other half of the magnet. The even orders of feed down from the two halves will add to each other. The measured coefficients in this case are given by:

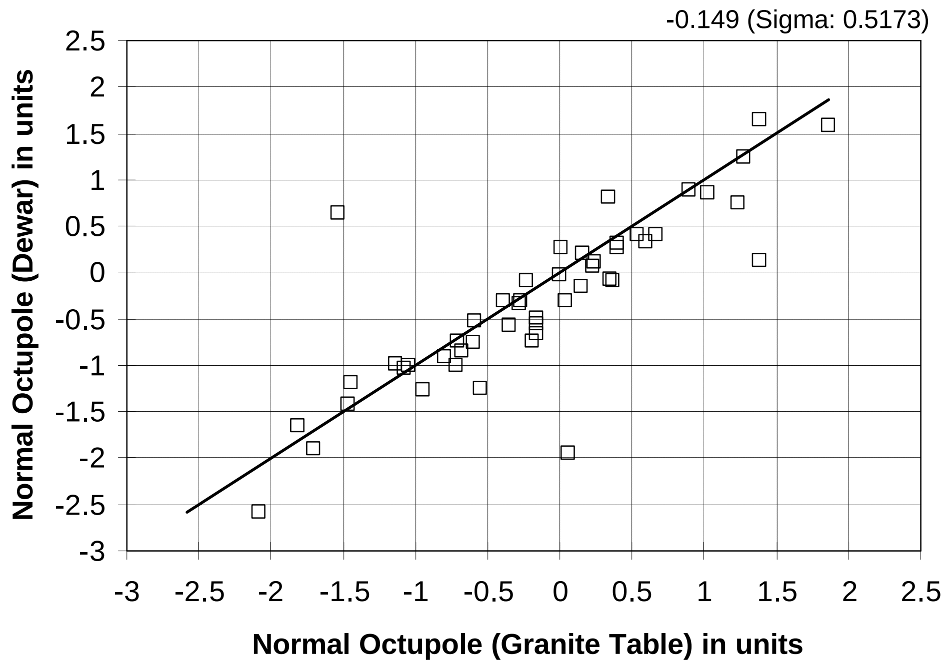
$$C'(n)\exp(-in\alpha'_n) = \sum_{\substack{k=n \\ (k-n)=\text{even}}}^{\infty} \frac{C(k)\exp(-ik\alpha_k)}{(k-n+1)} \frac{(k-1)!}{(n-1)!(k-n)!} \left(\frac{r_0 \exp(i\xi)}{R_{ref}} \right)^{k-n}$$

It should be noted that the summation includes only those values of k for which (k–n) is even. The lowest order correction term is of second order in (r_0/R_{ref}), and can be neglected in most cases for dipole and quadrupole magnets, since there can not be any second order feed down from the main field component. However, for sextupoles and magnets of higher multipolarity, there can be a second order feed down from the main harmonic, leading to large errors even with relatively small tilt. For example, the dipole field component will be incorrectly measured in a sextupole magnet, the quadrupole component in an octupole magnet, and so on.

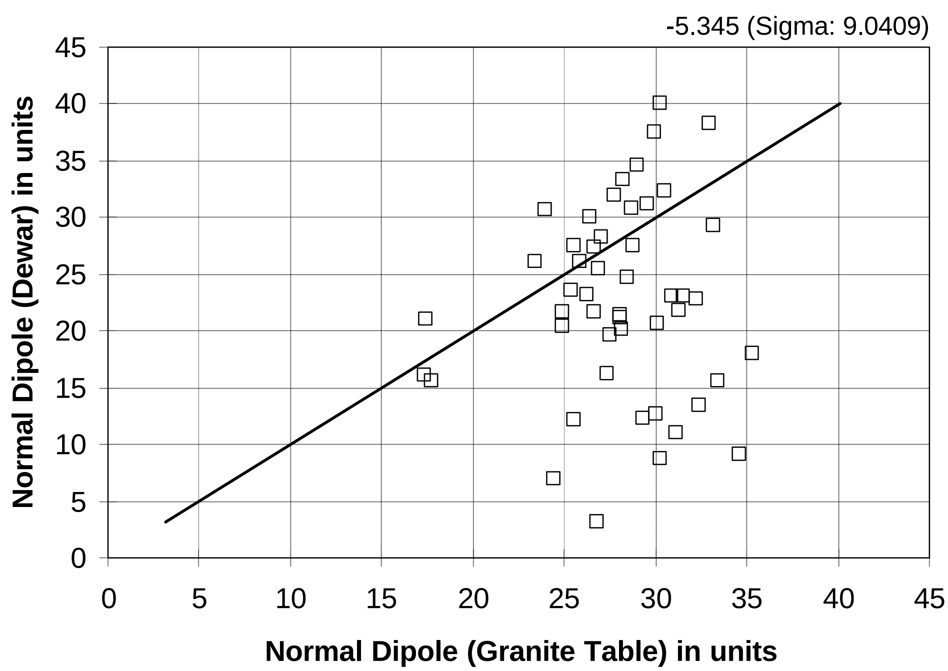
Often, magnets have rather large harmonics in the lead end region which are absent in the non-lead end region. In this case, even the first order terms from the two halves will not cancel each other, causing large errors. Examples of such errors are in the measurement of integral decapole terms in a quadrupole magnet having large dodecapole terms in the lead end region.

Effect of Tilt on Measurements of Sextupole Magnets

Granite Table Measurements: Measuring Coil Axis Parallel to Magnet Axis
Vertical Dewar Measurements: Measuring Coil Axis may not be Parallel.

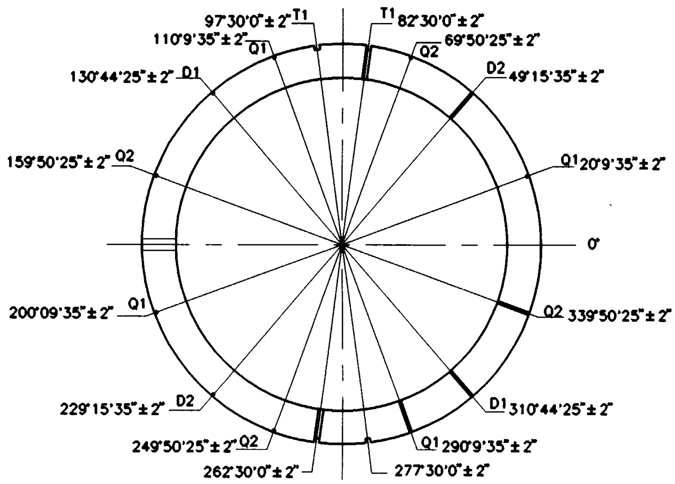


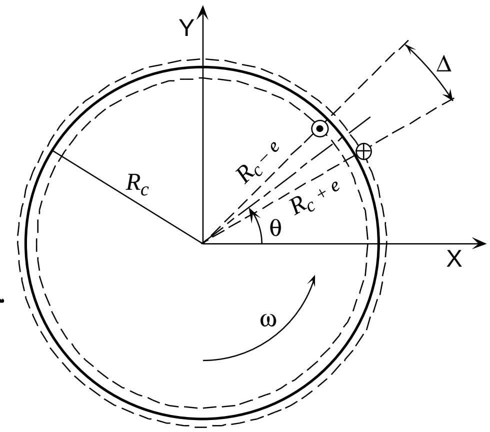
Octupole terms are not affected and show good correlation



Dipole terms are affected and show poor correlation

Calibration of a Five-Winding Tangential Coil





1. Dipole Buck Winding (DB1): R_1, δ_1
2. Dipole Buck Winding (DB2): R_2, δ_2
3. Tangential Winding (T): R_3, δ_3, Δ, ε
4. Quadrupole Buck Winding (QB1): R_4, δ_4
5. Quadrupole Buck Winding (QB2): R_5, δ_5

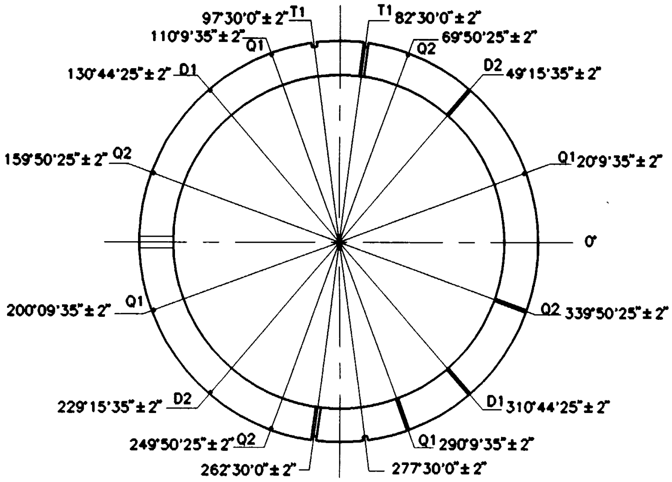
Total Number of Parameters required = 12.

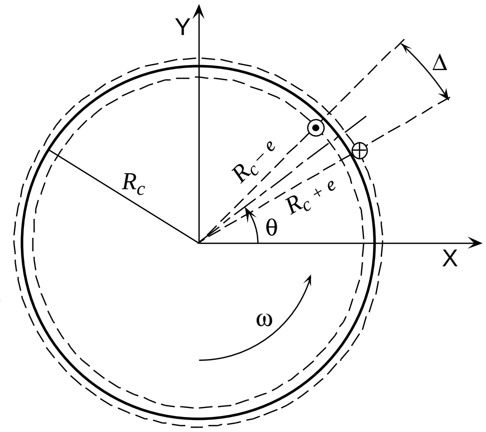
If a pure Dipole AND a pure Quadrupole magnet are available, with well defined phase angles, then the angle parameters for the four buck windings can be obtained without any difficulty. The measured angle of the tangential in the two fields may not be the same due to the tilt, ε. However, with well calibrated dipole and quadrupole fields, one could estimate the tilt using the expressions for the effect of a tilt.

- What if such calibrated magnets are not available ?
- How to get the various radii ?
- How to get the opening angle, Δ ?

It is possible to get all the angles relative to each other, all the radii relative to each other, the absolute value of the opening angle as well as the tangential tilt — all without using any knowledge of the field strength or direction !

Calibration of a Five-Winding Tangential Coil: Radii, D





Effect of a "Tilt" in the Tangential winding:

Amplitude Corr. Factor = $A_n = \sqrt{1+\left(\frac{n\varepsilon}{R_c}\right)^2 \cot^2\left(\frac{n\Delta}{2}\right)}$: NEGLECT (2ND ORDER)

Phase Correction $\lambda_n = \left(\frac{1}{n}\right)\tan^{-1}\left[\left(\frac{n\varepsilon}{R_c}\right)\cot\left(\frac{n\Delta}{2}\right)\right]$; $\delta_3(n) = \delta_3^0 + \lambda_n$

$V_j(n)$ = Amplitude of n-th harmonic in the j-th winding.

In a Dipole field, assuming a measuring coil longer than the magnet:

$$V_1(1) \propto N_1 R_1; \quad V_2(1) \propto N_2 R_2; \quad V_3(1) \propto N_3 R_3 \sin(\Delta/2)$$

$$\therefore \left(\frac{R_2}{R_1}\right) = \left(\frac{V_2(1)}{V_1(1)}\right)\left(\frac{N_1}{N_2}\right); \quad \left(\frac{R_3}{R_1}\right)\sin\left(\frac{\Delta}{2}\right) = \left(\frac{V_3(1)}{V_1(1)}\right)\left(\frac{N_1}{N_3}\right)$$

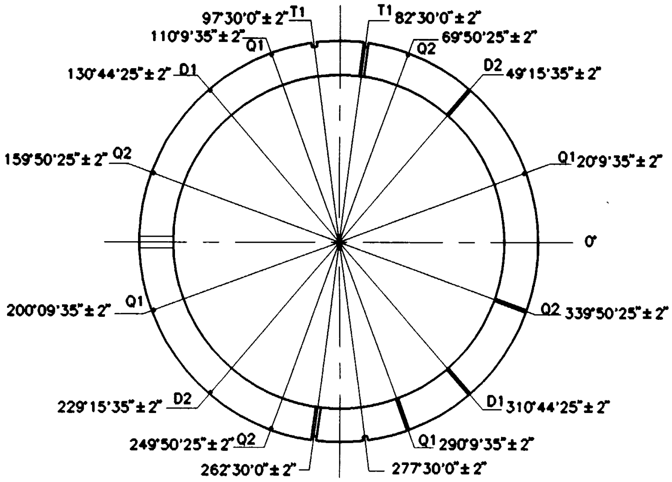
Similarly, in a quadrupole field:

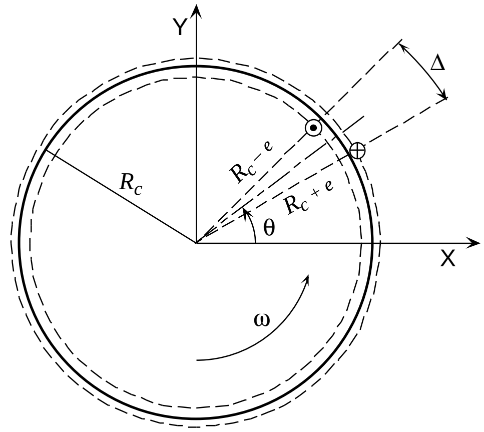
$$V_4(2) \propto 2N_4 R_4^2; \quad V_5(2) \propto 2N_5 R_5^2; \quad V_3(2) \propto N_3 R_3^2 \sin(\Delta)$$

$$\therefore \left(\frac{R_5}{R_4}\right) = \left[\left(\frac{V_5(2)}{V_4(2)}\right)\left(\frac{N_4}{N_5}\right)\right]^{1/2}; \quad \left(\frac{R_3}{R_4}\right)^2 \sin(\Delta) = \left(\frac{V_3(2)}{V_4(2)}\right)\left(\frac{2N_4}{N_3}\right)$$

If the field strengths in the dipole and the quadrupole magnets are also known, one can obtain the absolute values of R_1, R_2, R_4, R_5. From these values, the absolute values of R_3 and Δ can also be determined. If the absolute strengths are not known, one can only calculate the ratios of radii. Even then, we have only four equations in five unknowns and need more information.

Calibration of a Five-Winding Tangential Coil: Radii, D





$V_j(n)$ = Amplitude of n-th harmonic in the j-th winding.

From data in a Dipole field:

$$\left(\frac{R_2}{R_1}\right)=\left(\frac{V_2(1)}{V_1(1)}\right)\left(\frac{N_1}{N_2}\right);\quad \left(\frac{R_3}{R_1}\right)\sin\left(\frac{\Delta}{2}\right)=\left(\frac{V_3(1)}{V_1(1)}\right)\left(\frac{N_1}{N_3}\right)$$

From data in a Quadrupole field:

$$\left(\frac{R_5}{R_4}\right)=\left[\left(\frac{V_5(2)}{V_4(2)}\right)\left(\frac{N_4}{N_5}\right)\right]^{1/2};\quad \left(\frac{R_3}{R_4}\right)^2\sin(\Delta)=\left(\frac{V_3(2)}{V_4(2)}\right)\left(\frac{2N_4}{N_3}\right)$$

To know relative radii, five Unknowns: (R_2/R_1), (R_3/R_1), (R_4/R_1), (R_5/R_1), Δ

Need at least one more equation. If we use a sextupole field:

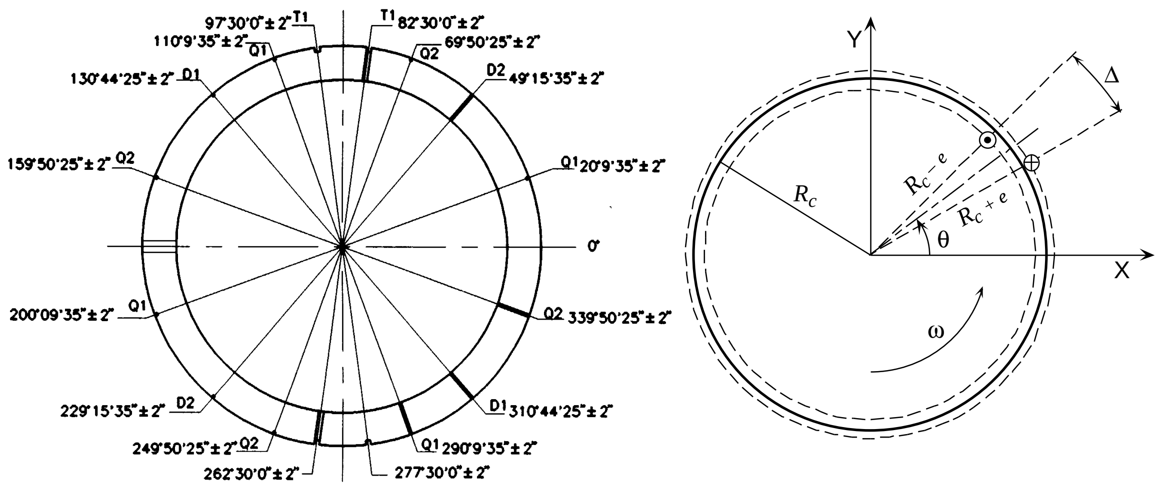
$$V_1(3)\propto N_1R_1^3;\quad V_2(3)\propto N_2R_2^3;\quad V_3(3)\propto N_3R_3^3\sin(3\Delta/2)$$

$$\therefore\left(\frac{R_2}{R_1}\right)=\left[\left(\frac{V_2(3)}{V_1(3)}\right)\left(\frac{N_1}{N_2}\right)\right]^{1/3};\quad \left(\frac{R_3}{R_1}\right)^3\sin(3\Delta/2)=\left(\frac{V_3(3)}{V_1(3)}\right)\left(\frac{N_1}{N_3}\right)$$

These equations give the five unknowns required to calculate the field strengths, namely, (R_2/R_1), (R_3/R_1), (R_4/R_1), (R_5/R_1) and Δ. We get one redundant equation, which could be used for consistency check on (R_2/R_1).

To know the absolute values of the radii, we need to determine just one radius. This could be obtained easily if a reference field is available. Otherwise, the value of R_1 (or any other winding) can be simply guessed from mechanical measurements, or calibrated against other known coils.

Calibration of a Five-Winding Tangential Coil: Angles and e



If reference dipole and quadrupole magnets are available with precisely known phase angles, the angles δ_1, δ_2, δ_4, and δ_5 can be easily determined from the phases of the dipole and the quadrupole components of the measured signals. Also, we can get $\delta_3(1)$ in a dipole magnet and $\delta_3(2)$ in a quadrupole magnet. These can be used to obtain δ_3^0 and (ε/R_3) using the relations:

$$\delta_3(n) = \delta_3^0 + \lambda_n\,; \qquad \lambda_n = \left(\frac{1}{n}\right)\tan^{-1}\left[\left(\frac{n\varepsilon}{R_c}\right)\cot\left(\frac{n\Delta}{2}\right)\right]$$

If the absolute phase angles of both the dipole and the quadrupole fields are not known, we make use of a sextupole field also. We can determine:

In Dipole field: $\delta_2 - \delta_1$ and $\delta_3(1) - \delta_1$
In Quadrupole field: $\delta_5 - \delta_4$ and $\delta_3(2) - \delta_4$
In Sextupole field: $\delta_2 - \delta_1$ and $\delta_3(3) - \delta_1$

Combining the data from the dipole and the sextupole fields, we get the quantity $\delta_3(3) - \delta_3(1)$, which depends on (ε/R_3) and the opening angle, Δ. Since Δ is obtained from calibration of the radii, the parameter (ε/R_3) is determined. Knowing (ε/R_3), one can calculate $\delta_3(2)$–δ_1, which can be combined with the data in a quadrupole field to get δ_4 and δ_5 also relative to δ_1. All angles are thus known relative to one of the windings. For measuring coils equipped with a gravity sensor, the absolute angles can be obtained by making measurements from the lead and the non-lead ends of a magnet. For other systems, absolute values of angles are often unnecessary.