Multipole Expansion of Two Dimensional Field in Free Space

In free space (no true currents), the magnetic field **H** (= **B**/μ_0) satisfies:

$$\nabla \times \mathbf{H} = 0 \quad \Rightarrow \quad \nabla \times \mathbf{B} = 0 \Rightarrow \quad \mathbf{B} = -\nabla \Phi_m ,$$

Φ_m = MAGNETIC SCALAR POTENTIAL.

The magnetic induction, $\mathbf{B} = \mu_0 \mathbf{H}$ always satisfies $\nabla . \mathbf{B} = 0$. Therefore,

$$\nabla^2 \Phi_m = 0$$

LAPLACE'S EQN. FOR SCALAR POTENTIAL

In cylindrical coordinates, and for no z-dependence (2-D fields):

$$\left(\frac{1}{r}\right)\frac{\partial}{\partial r}\left(r\frac{\partial \Phi_m}{\partial r}\right) + \left(\frac{1}{r^2}\right)\left(\frac{\partial^2 \Phi_m}{\partial \theta^2}\right) = 0$$

writing $\Phi_m(r,\theta) = R(r)\Theta(\theta)$, and imposing the conditions

$$\Theta(\theta + 2\pi) = \Theta(\theta); \quad R(r) = \text{ finite at } r = 0$$

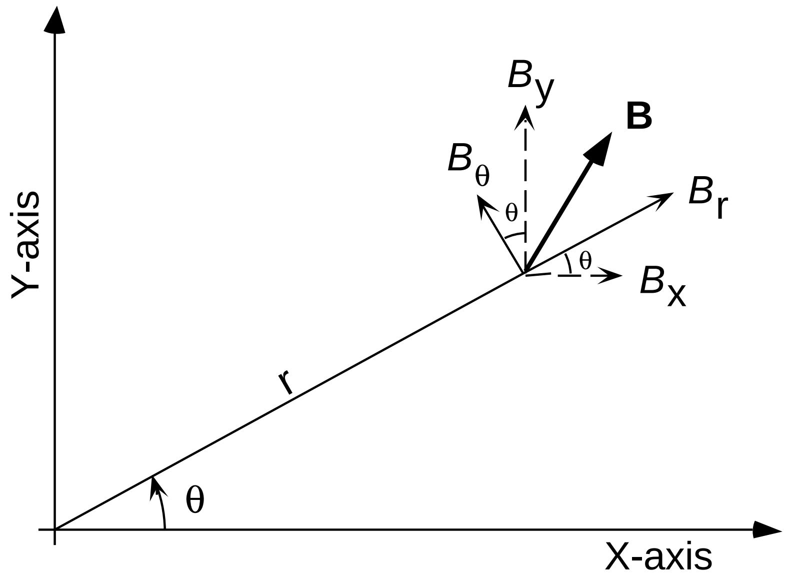
we get the solution of the Laplace's equation:

$$B_r(r,\theta) = -\left(\frac{\partial \Phi_m}{\partial r}\right) = \sum_{n=1}^{\infty} C(n)\left(\frac{r}{R_{ref}}\right)^{n-1} \sin[n(\theta - \alpha_n)]$$

$$B_\theta(r,\theta) = -\left(\frac{1}{r}\right)\left(\frac{\partial \Phi_m}{\partial \theta}\right) = \sum_{n=1}^{\infty} C(n)\left(\frac{r}{R_{ref}}\right)^{n-1} \cos[n(\theta - \alpha_n)]$$

$C(n)$ and α_n are constants and R_{ref} is an arbitrary REFERENCE RADIUS, typically chosen to be 50-70% of the magnet aperture.

Multipole Expansion of Two Dimensional Field in Free Space



From the cylindrical components, we can get the Cartesian components:

$$B_x(r,\theta) = B_r \cos\theta - B_\theta \sin\theta = \sum_{n=1}^{\infty} C(n)\left(\frac{r}{R_{ref}}\right)^{n-1} \sin[(n-1)\theta - n\alpha_n]$$

$$B_y(r,\theta) = B_r \sin\theta + B_\theta \cos\theta = \sum_{n=1}^{\infty} C(n)\left(\frac{r}{R_{ref}}\right)^{n-1} \cos[(n-1)\theta - n\alpha_n]$$

Define a Complex function, $\boldsymbol{B}(\boldsymbol{z}) = B_y(x,\ y) + iB_x(x,\ y)$ of the complex variable, $\boldsymbol{z} = x + iy = r.\exp(i\theta)$:

$$\boldsymbol{B}(\boldsymbol{z}) = B_y(x,y) + iB_x(x,y) = \sum_{n=1}^{\infty} \left[C(n)\exp(-ina_n)\right]\left(\frac{\boldsymbol{z}}{R_{ref}}\right)^{n-1}$$

Normal and Skew Components of the Two-dimensional Field

The complex field $\boldsymbol{B}(\boldsymbol{z})$ is given by:

$$\boldsymbol{B}(\boldsymbol{z}) = B_y(x,y) + iB_x(x,y) = \sum_{n=1}^{\infty} \left[C(n)\exp(-in\alpha_n)\right]\left(\frac{\boldsymbol{z}}{R_{ref}}\right)^{n-1}$$

The real and the imaginary parts of the expansion coefficients are defined as the NORMAL and the SKEW Components:

$$C(n)\exp(-in\alpha_n) = (2n\text{-POLE NORMAL TERM}) + i(2n\text{-POLE SKEW TERM})$$

Unfortunately, the index n in the expansion coefficient is not the same as the corresponding power (n–1) of $\boldsymbol{z}$ in the above equation. This has led to two different conventions in denoting the normal and the skew terms:

$$\begin{aligned} 2n\text{-pole Normal Term} &= C(n)\cos(n\alpha_n) \equiv B_{n-1} \\ 2n\text{-pole Skew Term} &= -C(n)\sin(n\alpha_n) \equiv A_{n-1} \end{aligned} \qquad \text{“U.S. CONVENTION”}$$

$$\begin{aligned} 2n\text{-pole Normal Term} &= C(n)\cos(n\alpha_n) \equiv B_{n} \\ 2n\text{-pole Skew Term} &= -C(n)\sin(n\alpha_n) \equiv A_{n} \end{aligned} \qquad \text{“EUROPEAN CONVENTION”}$$

In terms of the normal and skew components, the expansion of $\boldsymbol{B}(\boldsymbol{z})$ is:

$$\boldsymbol{B}(\boldsymbol{z}) = B_y + iB_x = \sum_{n=0}^{\infty}\left[B_n + iA_n\right]\left(\frac{\boldsymbol{z}}{R_{ref}}\right)^{n} \qquad \text{“U.S. CONVENTION”}$$

$$\boldsymbol{B}(\boldsymbol{z}) = B_y + iB_x = \sum_{n=1}^{\infty}\left[B_n + iA_n\right]\left(\frac{\boldsymbol{z}}{R_{ref}}\right)^{n-1} \qquad \text{“EUROPEAN CONVENTION”}$$

The “U.S. Convention” leads to a more elegant looking equation without the (n–1) in the powers, whereas the “European Convention” retains the simple relationship between the n-th term and the number of poles it represents.

For work done in the US for the LHC project, the "European Convention" is generally followed.

Normal and Skew Components as Field Derivatives

The complex field is given by:

$$\boldsymbol{B}(\boldsymbol{z}) = B_y + iB_x = \sum_{n=0}^{\infty} \left[B_n + iA_n \right] \left(\frac{\boldsymbol{z}}{R_{ref}} \right)^n \qquad \text{“U.S. CONVENTION”}$$

$$\boldsymbol{B}(\boldsymbol{z}) = B_y + iB_x = \sum_{n=1}^{\infty} \left[B_n + iA_n \right] \left(\frac{\boldsymbol{z}}{R_{ref}} \right)^{n-1} \qquad \text{“EUROPEAN CONVENTION”}$$

where B_n and A_n are the Normal and the Skew Components.

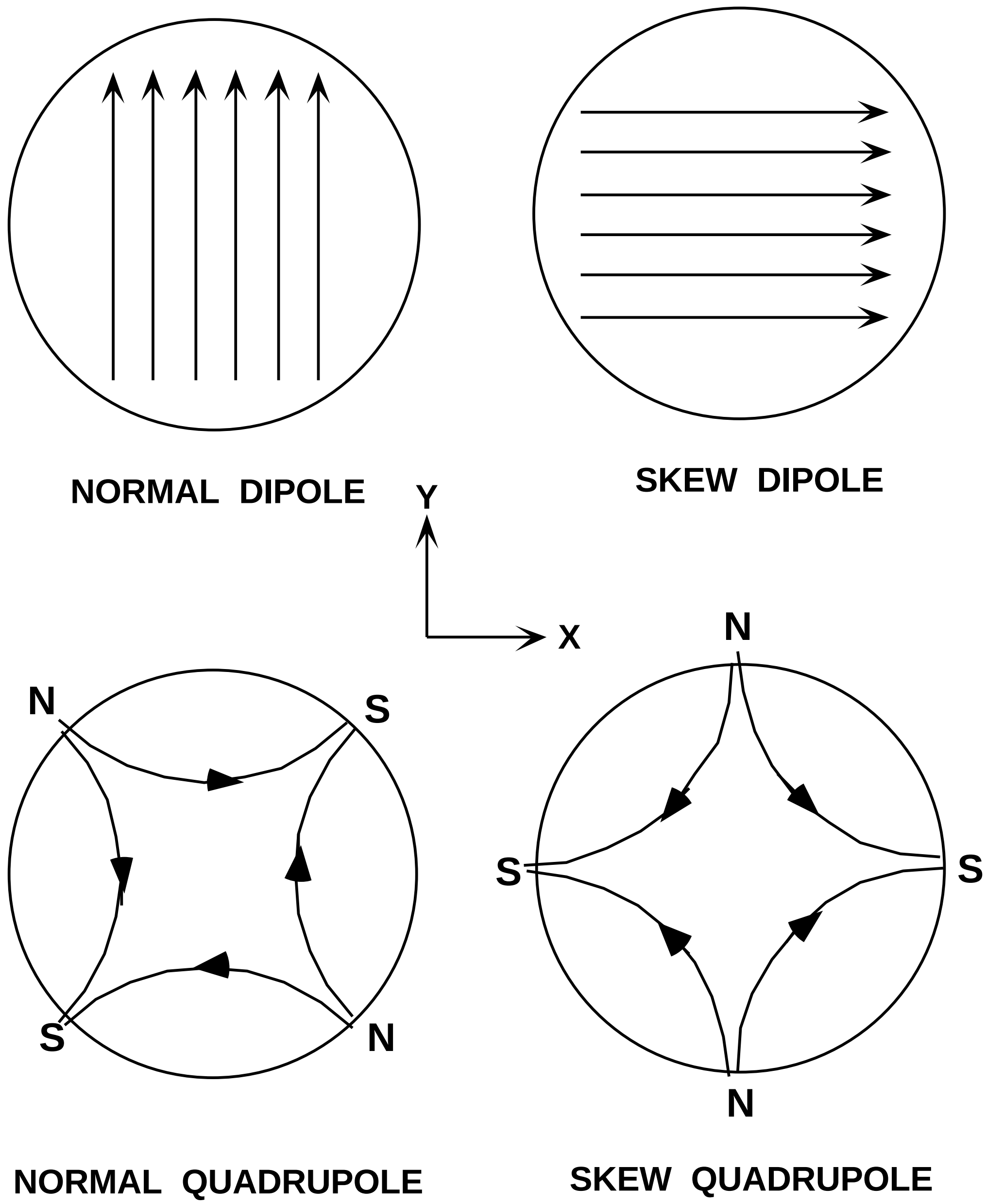
It is possible to assign a physical significance to the normal and the skew components in terms of the gradients of the field. It is easy to show that

$$B_n(\mathrm{US}) = B_{n+1}(\mathrm{European}) = \frac{R_{ref}^n}{n!} \left(\frac{\partial^n B_y}{\partial x^n} \right) \Bigg|_{x=0;\, y=0}$$

$$A_n(\mathrm{US}) = A_{n+1}(\mathrm{European}) = \frac{R_{ref}^n}{n!} \left(\frac{\partial^n B_x}{\partial x^n} \right) \Bigg|_{x=0;\, y=0} \qquad n \geq 0$$

The *normal* components of various orders are thus related to the derivatives of corresponding orders of the *vertical* component of the field along the midplane of the magnet. Similarly, the *skew* components of various orders are related to the derivatives of corresponding orders of the *horizontal* component of the field along the midplane of the magnet.

Examples of Normal and Skew Magnets



Fractional Field Coefficients or Multipoles

The expansion of the complex field is given by:

$$\boldsymbol{B}(\boldsymbol{z}) = B_y(x,y) + iB_x(x,y) = \sum_{n=1}^{\infty}\left[C(n)\exp(-in\alpha_n)\right]\left(\frac{\boldsymbol{z}}{R_{ref}}\right)^{n-1}$$

The real and the imaginary parts of the expansion coefficients are referred to as the Normal and the Skew components respectively. These expansion coefficients lead to the actual field strength in the magnet and are dependent on the excitation level of a magnet. In accelerator applications, one is often interested in the *shape* of the field, rather than its absolute magnitude. This is done by expressing the various harmonic terms in the expansion as a fraction of a REFERENCE FIELD, B_{ref} :

$$\boldsymbol{B}(\boldsymbol{z}) = B_y(x,y) + iB_x(x,y) = B_{ref}\sum_{n=1}^{\infty}\left[\frac{C(n)\exp(-in\alpha_n)}{B_{ref}}\right]\left(\frac{\boldsymbol{z}}{R_{ref}}\right)^{n-1}$$

Generally, this reference field is chosen as the strength of the most dominant term in the expansion. For a $2m$-pole magnet, it is expected that the term for $n = m$ will be the most dominant one. Hence, B_{ref} may be chosen to be equal to $C(m)$.

The Normal and Skew $2n$-pole fractional field coefficients, or "MULTIPOLES" are defined as:

$$b_{n-1} = \mathrm{Re}\left[\frac{C(n)\exp(-in\alpha_n)}{B_{ref}}\right] = \frac{B_{n-1}}{B_{ref}}; \quad a_{n-1} = \mathrm{Im}\left[\frac{C(n)\exp(-in\alpha_n)}{B_{ref}}\right] = \frac{A_{n-1}}{B_{ref}} \qquad \text{(US)}$$

$$b_n = \mathrm{Re}\left[\frac{C(n)\exp(-in\alpha_n)}{B_{ref}}\right] = \frac{B_n}{B_{ref}}; \; a_n = \mathrm{Im}\left[\frac{C(n)\exp(-in\alpha_n)}{B_{ref}}\right] = \frac{A_n}{B_{ref}} \qquad \text{(EUROPEAN)}$$

In a typical accelerator magnet, these coefficients are of the order of 10^{-4}. The coefficients are therefore often quoted after multiplying by 10^4. With this multiplicative factor, the values of the multipoles are said to be in "units".

Analyticity of the Complex Field $B(z)$

Any function of complex variable $\boldsymbol{z}$ given by:

$$\boldsymbol{F}(\boldsymbol{z}) = U(x, y) + i\, V(x, y)$$

is an Analytic Function of $\boldsymbol{z}$ if the real and the imaginary parts of the function satisfy the Cauchy-Riemann conditions:

$$\left(\frac{\partial U}{\partial x}\right) = \left(\frac{\partial V}{\partial y}\right) \quad \text{and} \quad \left(\frac{\partial U}{\partial y}\right) = -\left(\frac{\partial V}{\partial x}\right) \qquad \text{CAUCHY-RIEMANN CONDITIONS}$$

An analytic function of $\boldsymbol{z}$ can be shown to be a function of $\boldsymbol{z}$ alone, and not its complex conjugate, $\boldsymbol{z^*}$. For the case of the 2-dimensional field in a source free region, Maxwell's equations give:

$$\nabla.\mathbf{B} = 0 \Rightarrow \left(\frac{\partial B_x}{\partial x}\right) + \left(\frac{\partial B_y}{\partial y}\right) = 0; \;\; \text{or} \;\; \left(\frac{\partial B_y}{\partial y}\right) = -\left(\frac{\partial B_x}{\partial x}\right)$$

$$(\nabla \times \mathbf{B})_z = 0 \Rightarrow \left(\frac{\partial B_y}{\partial x}\right) - \left(\frac{\partial B_x}{\partial y}\right) = 0; \;\; \text{or} \;\; \left(\frac{\partial B_y}{\partial x}\right) = \left(\frac{\partial B_x}{\partial y}\right)$$

The Maxwell's equations are identical to the Cauchy-Riemann conditions if we choose $U(x, y) = B_y(x, y)$ and $V(x, y) = B_x(x, y)$. Thus the function $\boldsymbol{B}(\boldsymbol{z}) = B_y(x, y) + i\, B_x(x, y)$ defined earlier is an analytic function of the complex variable, $\boldsymbol{z}$. It should be noted that the function

$$B_x(x, y) - iB_y(x, y) \qquad \text{IS ALSO AN ANALYTIC FUNCTION OF } \boldsymbol{z}$$

However,

$$B_x(x, y) + iB_y(x, y) \qquad \text{IS NOT AN ANALYTIC FUNCTION OF } \boldsymbol{z}$$

An analytic function of $\boldsymbol{z}$ can be expressed as a power series in $\boldsymbol{z}$, as we have already seen for the case of $\boldsymbol{B}(\boldsymbol{z}) = B_y(x, y) + i\, B_x(x, y)$. The analyticity is useful in dealing with two dimensional problems in magnetostatics.

The Complex Potential

We define a complex field, $\boldsymbol{B}(\boldsymbol{z})$ as:

$$\boldsymbol{B}(\boldsymbol{z}) = B_y(x,y) + iB_x(x,y) = \sum_{n=1}^{\infty}\left[C(n)\exp(-in\alpha_n)\right]\left(\frac{\boldsymbol{z}}{R_{ref}}\right)^{n-1}$$

which is an analytic function of the complex variable $\boldsymbol{z} = x + iy$. Accordingly, we define a COMPLEX POTENTIAL $\boldsymbol{W}(\boldsymbol{z})$ such that:

$$\boldsymbol{B}(\boldsymbol{z}) = -\frac{d\boldsymbol{W}(\boldsymbol{z})}{d\boldsymbol{z}}$$

It can be shown that the real and imaginary parts of this complex potential are nothing but the vector and the scalar potentials respectively:

$$\boldsymbol{W}(\boldsymbol{z}) = W_r(x,y) + iW_i(x,y)$$

$$\frac{d\boldsymbol{W}(\boldsymbol{z})}{dx} = \frac{d\boldsymbol{W}(\boldsymbol{z})}{d\boldsymbol{z}}.\frac{d\boldsymbol{z}}{dx} = -(B_y + iB_x).1 = \left(\frac{\partial W_r}{\partial x}\right) + i\left(\frac{\partial W_i}{\partial x}\right)$$

$$\therefore \left(\frac{\partial W_r}{\partial x}\right) = -B_y = \left(\frac{\partial A_z}{\partial x}\right); \quad \left(\frac{\partial W_i}{\partial x}\right) = -B_x = \left(\frac{\partial \Phi_m}{\partial x}\right)$$

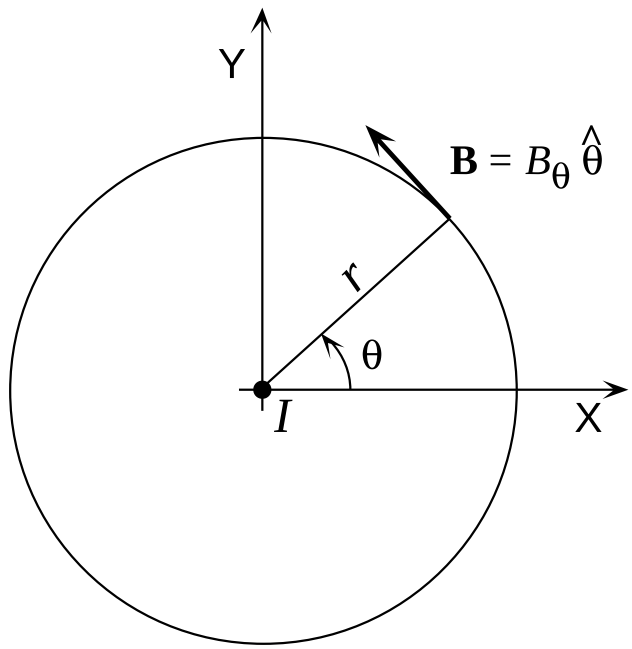
Similarly,

$$\frac{d\boldsymbol{W}(\boldsymbol{z})}{dy} = \frac{d\boldsymbol{W}(\boldsymbol{z})}{d\boldsymbol{z}}.\frac{d\boldsymbol{z}}{dy} = -(B_y + iB_x).i = \left(\frac{\partial W_r}{\partial y}\right) + i\left(\frac{\partial W_i}{\partial y}\right)$$

$$\therefore \left(\frac{\partial W_r}{\partial y}\right) = B_x = \left(\frac{\partial A_z}{\partial y}\right); \quad \left(\frac{\partial W_i}{\partial y}\right) = -B_y = \left(\frac{\partial \Phi_m}{\partial y}\right)$$

It is clear from the above that the real part of the complex potential is the (only) component, A_z, of the vector potential **A**, and the imaginary part of the complex potential is the magnetic scalar potential, Φ_m. The complex potential therefore contains a complete description of any field problem.

Complex Field due to a Current Filament at the Origin



We first consider the simplest case of an infinitely long current filament located at the origin, carrying a current I along the positive Z-axis, as shown in the figure. At any point P located at (r, θ), the magnetic field has the same magnitude along a circle of radius r and is directed along the azimuthal direction. From Ampère's law:

$$\mathbf{B} = \mu_0 \mathbf{H} = \frac{\mu_0 I}{2\pi r}\hat{\theta}$$

$$B_x = B_r \cos\theta - B_\theta \sin\theta = -\frac{\mu_0 I}{2\pi r}\sin\theta; \; B_y = B_r \sin\theta + B_\theta \cos\theta = \frac{\mu_0 I}{2\pi r}\cos\theta$$

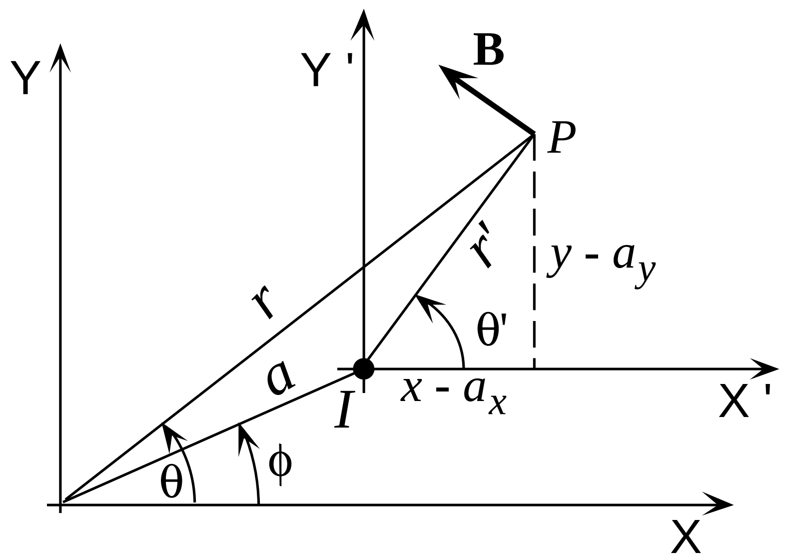
$$\boldsymbol{B}(\boldsymbol{z}) = B_y + iB_x = \frac{\mu_0 I}{2\pi r.\exp(i\theta)} = \frac{\mu_0 I}{2\pi \boldsymbol{z}}$$

The Complex Potential is therefore given by:

$$\boldsymbol{W}(\boldsymbol{z}) = -\int B(\boldsymbol{z})d\boldsymbol{z} = -\left(\frac{\mu_0 I}{2\pi}\right)\ln(\boldsymbol{z}) + \text{constant} = -\left(\frac{\mu_0 I}{2\pi}\right)[\ln(r) + i\theta] + \text{constant}$$

The real and the imaginary parts of $\boldsymbol{W}(\boldsymbol{z})$ give the vector potential, A_z, and the scalar potential, Φ_m, respectively.

Current Filament at an Arbitrary Location



Let us consider the magnetic induction **B** at a point $P(r, \theta)$ due to an infinitely long current filament located at the point $\boldsymbol{a} = a\exp(i\phi) = a_x + ia_y$. In a frame X$'$-Y$'$ with axes parallel to X-Y and having origin at the current filament, the complex field is given by

$$\boldsymbol{B}(\boldsymbol{z}') = B_{y'} + iB_{x'} = \left(\frac{\mu_o I}{2\pi \boldsymbol{z}'}\right)$$

$$\therefore \boldsymbol{B}(\boldsymbol{z}) = B_y + iB_x = B_{y'} + iB_{x'} = \boldsymbol{B}(\boldsymbol{z}') = \left(\frac{\mu_o I}{2\pi \boldsymbol{z}'}\right) = \left(\frac{\mu_o I}{2\pi(\boldsymbol{z} - \boldsymbol{a})}\right)$$

The complex potential is given by:

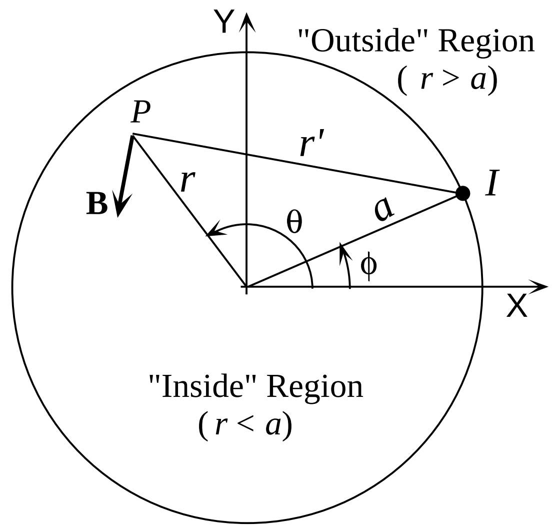
$$\boldsymbol{W}(\boldsymbol{z}) = -\int \boldsymbol{B}(\boldsymbol{z})d\boldsymbol{z} = -\left(\frac{\mu_0 I}{2\pi}\right)\ln(\boldsymbol{z} - \boldsymbol{a}) + \text{const.} = -\left(\frac{\mu_0 I}{2\pi}\right)[\ln(r') + i\theta'] + \text{const.}$$

The real and the imaginary parts of the complex potential give the component, A_z , of the vector potential and the magnetic scalar potential, Φ_m, respectively. Knowing the field due to a current filament at an arbitrary location, we can determine the field due to any arbitrary distribution of current by superposition.

Multipole Expansion of Field due to a Current Filament

The complex field at any point P due to a current filament at $\boldsymbol{z} = \boldsymbol{a}$ is given by:

$$\boldsymbol{B}(\boldsymbol{z}) = \left(\frac{\mu_0 I}{2\pi}\right) . \frac{1}{(\boldsymbol{z}-\boldsymbol{a})}$$



This expression is valid for all values of $\boldsymbol{z}$ in the complex plane and is singular at $\boldsymbol{z} = \boldsymbol{a}$. We are interested in a multipole expansion of the complex field in the form of a power series :

$$\boldsymbol{B}(\boldsymbol{z}) = \sum_{n=1}^{\infty} \left[C(n) \exp(-in\alpha_n)\right] \left(\frac{\boldsymbol{z}}{R_{ref}}\right)^{n-1}$$

Clearly, such an expansion can not be valid for all values of $\boldsymbol{z}$. The complex field has a singularity at $\boldsymbol{z} = \boldsymbol{a}$ and the series diverges for $|\boldsymbol{z}| \to \infty$. Let us divide the entire X-Y plane into two regions – an **"INSIDE"** region extending to $r < a$ and an **"OUTSIDE"** region for $r > a$. The general expression for the complex field can be expanded as a different power series in each of these two regions. Let us consider the "inside" region first.

Inside Region ($r < a$):

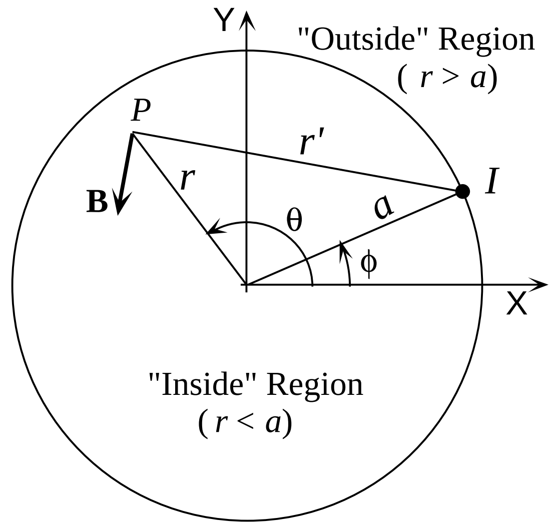
$$\boldsymbol{B}_{in}(\boldsymbol{z}) = \left(\frac{\mu_0 I}{2\pi}\right) . (\boldsymbol{z}-\boldsymbol{a})^{-1} = -\left(\frac{\mu_0 I}{2\pi a \exp(i\phi)}\right) \left[1 - \left(\frac{r}{a}\right) \exp\{i(\theta - \phi)\}\right]^{-1}$$

Using the binomial expansion $(1-\xi)^{-1} = \sum_{n=1}^{\infty} \xi^{n-1}$, we get,

$$\boldsymbol{B}_{in}(\boldsymbol{z}) = -\left(\frac{\mu_0 I}{2\pi a}\right) \sum_{n=1}^{\infty} \exp(-in\phi) \left(\frac{R_{ref}}{a}\right)^{n-1} \left(\frac{\boldsymbol{z}}{R_{ref}}\right)^{n-1}$$

which is in the desired form.

Multipole Expansion of Field due to a Current Filament



The complex field in the **"INSIDE"** region is given by:

$$\boldsymbol{B}_{in}(\boldsymbol{z}) = -\left(\frac{\mu_0 I}{2\pi a}\right)\sum_{n=1}^{\infty} \exp(-in\phi)\left(\frac{R_{ref}}{a}\right)^{n-1}\left(\frac{\boldsymbol{z}}{R_{ref}}\right)^{n-1}$$

$$\equiv \sum_{n=1}^{\infty}[C(n)\exp(-i\alpha_n)].\left(\frac{\boldsymbol{z}}{R_{ref}}\right)^{n-1}$$

$$C(n) = \left|\frac{\mu_0 I}{2\pi a}\right|.\left(\frac{R_{ref}}{a}\right)^{n-1}; \quad \alpha_n = \phi + \frac{\pi}{n} \ \text{ for } I > 0\,; \quad \alpha_n = \phi \ \text{ for } I < 0$$

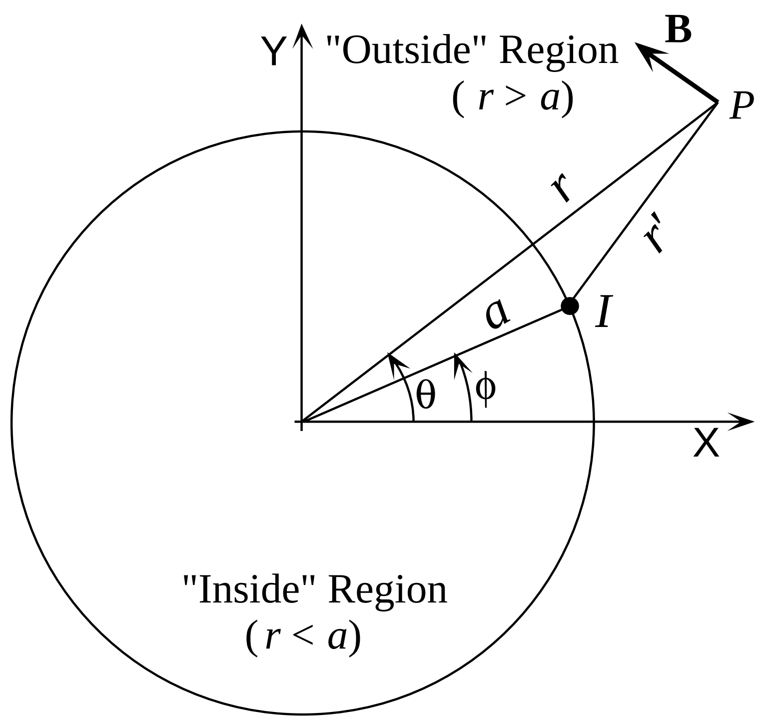
The NORMAL and SKEW components of the $2n$-pole field are (in US notation):

$$B_{n-1} = -\left(\frac{\mu_0 I}{2\pi a}\right).\left(\frac{R_{ref}}{a}\right)^{n-1}\cos(n\phi); \quad A_{n-1} = \left(\frac{\mu_0 I}{2\pi a}\right).\left(\frac{R_{ref}}{a}\right)^{n-1}\sin(n\phi)$$

In the European notation, the $2n$-pole components are denoted by:

$$B_n = -\left(\frac{\mu_0 I}{2\pi a}\right).\left(\frac{R_{ref}}{a}\right)^{n-1}\cos(n\phi); \quad A_n = \left(\frac{\mu_0 I}{2\pi a}\right).\left(\frac{R_{ref}}{a}\right)^{n-1}\sin(n\phi)$$

Multipole Expansion of Field due to a Current Filament



For the **"OUTSIDE"** Region, we can expand the complex field as:

$$\boldsymbol{B}_{out}(\boldsymbol{z}) = \left(\frac{\mu_0 I}{2\pi}\right).(\boldsymbol{z}-\boldsymbol{a})^{-1} = \left(\frac{\mu_0 I}{2\pi r \exp(i\theta)}\right)\left[1-\left(\frac{a}{r}\right)\exp\{i(\phi-\theta)\}\right]^{-1}$$

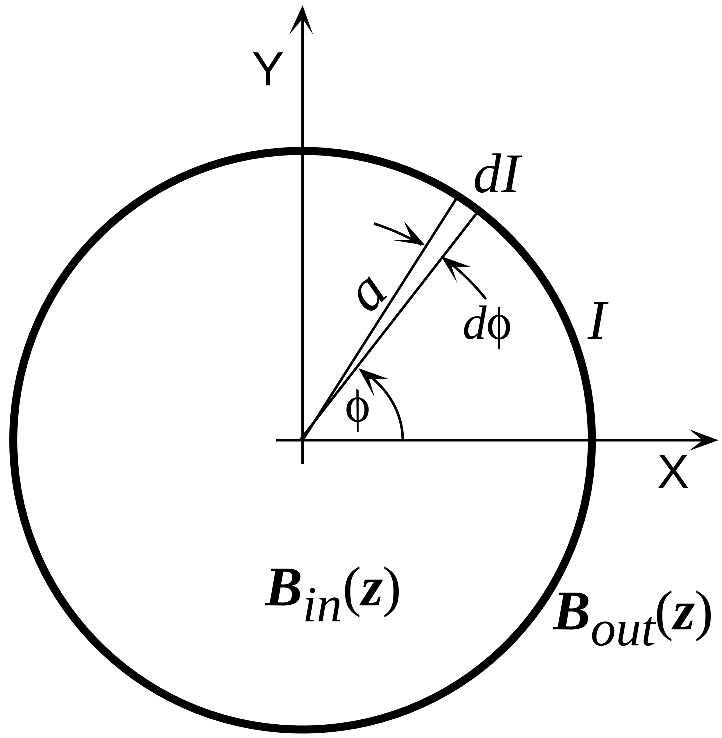
We use the binomial expansion $(1-\xi)^{-1} = 1+\sum_{n=1}^{\infty}\xi^n$ to get,

$$\boldsymbol{B}_{out}(\boldsymbol{z}) = \left(\frac{\mu_0 I}{2\pi \boldsymbol{z}}\right)\left[1+\sum_{n=1}^{\infty}[\cos(n\phi)+i\sin(n\phi)]\left(\frac{a}{\boldsymbol{z}}\right)^n\right]$$

This series is **NOT** in the usual form of the multipole expansion for the field inside a magnet aperture. However, it can be seen that this expansion converges for $|\boldsymbol{z}|\to\infty$ to $\boldsymbol{B}_{out}(\boldsymbol{z}) \to [\mu_0 I/(2\pi\boldsymbol{z})]$ as is expected.

Cylindrical Current Sheet of Uniform Current Density

Let us consider a cylindrical current sheet of radius a carrying a total current of I. The current is assumed to be distributed uniformly and flowing only along the Z-direction.



The complex field at any point can be obtained by integrating the contributions from small elements of width $d\phi$, as shown in the figure. The expansion of field from a filament for $r < a$ is different from that for $r > a$..

The field due to the current element $dI = (I.d\phi)/(2\pi)$ at any point inside is:

$$d\boldsymbol{B}_{in}(\boldsymbol{z}) = -\left(\frac{\mu_0}{2\pi a}\right)\left(\frac{I}{2\pi}\right)\sum_{n=1}^{\infty}\left(\frac{\boldsymbol{z}}{a}\right)^{n-1}\exp(-in\phi)\,d\phi$$

$$\therefore\ \boldsymbol{B}_{in}(\boldsymbol{z}) = -\left(\frac{\mu_0}{2\pi a}\right)\left(\frac{I}{2\pi}\right)\sum_{n=1}^{\infty}\left(\frac{\boldsymbol{z}}{a}\right)^{n-1}\int_0^{2\pi}\exp(-in\phi)\,d\phi = 0$$

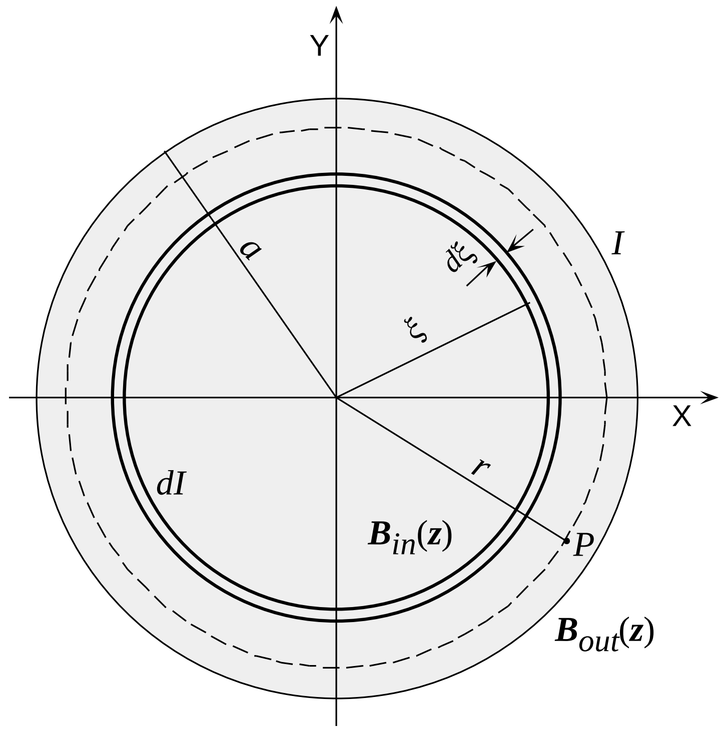
Similary, for any point outside the cylindrical shell, we have

$$\boldsymbol{B}_{out}(\boldsymbol{z}) = \left(\frac{\mu_0}{2\pi\boldsymbol{z}}\right)\left(\frac{I}{2\pi}\right)\left[\int_0^{2\pi}d\phi + \sum_{n=1}^{\infty}\left(\frac{a}{\boldsymbol{z}}\right)^{n}\int_0^{2\pi}\exp(in\phi)\,d\phi\right] = \left(\frac{\mu_0 I}{2\pi\boldsymbol{z}}\right)$$

Thus, for points outside, a cylindrical shell of uniform current density behaves as a current filament located at the origin. The field is zero inside the cylinder.

Solid Cylindrical Conductor with Uniform Current Density

We consider a solid cylinder of radius a carrying a total current *I*. The field at any point can be obtained by dividing the solid cylinder into thin shells. For any point outside the cylindrical conductor, all such shells contribute. For any point *P* at a radius $r < a$, as shown in the figure, only shells with radii $x < r$ contribute.



Total current carried by a thin shell of radius *x* and thickness *dx* is

$$dI = J.2\pi\xi d\xi; \quad \text{where} \quad J = \frac{I}{\pi a^2} \quad \text{is the current density}$$

For any point inside the cylinder, the total complex field is:

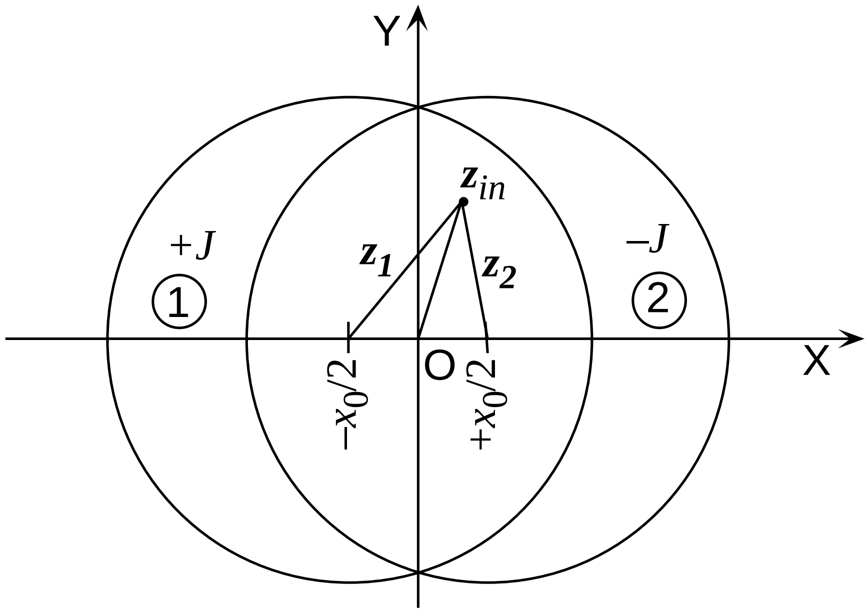
$$\boldsymbol{B}_{in}(\boldsymbol{z}) = \int \left(\frac{\mu_o dI}{2\pi \boldsymbol{z}} \right) = \left(\frac{\mu_o J}{\boldsymbol{z}} \right) . \int_0^r \xi . d\xi = \left(\frac{\mu_o J r^2}{2\boldsymbol{z}} \right) = \left(\frac{\mu_o J}{2} \right) \boldsymbol{z}^*$$

where we have used the fact that $r^2 = \boldsymbol{z}\boldsymbol{z}^*$. For any point outside the solid cylinder, the total complex field is:

$$\boldsymbol{B}_{out}(\boldsymbol{z}) = \int \left(\frac{\mu_o dI}{2\pi \boldsymbol{z}} \right) = \left(\frac{\mu_o J}{\boldsymbol{z}} \right) . \int_0^a \xi . d\xi = \left(\frac{\mu_o J a^2}{2\boldsymbol{z}} \right) = \left(\frac{\mu_o J}{2} \right) \left(\frac{a^2}{\boldsymbol{z}} \right)$$

It should be noted that $\boldsymbol{B}_{in}(\boldsymbol{z})$ is a function of $\boldsymbol{z}^*$, and hence is **NOT** an analytic function of $\boldsymbol{z}$.

Two Overlapping Cylinders: Pure Dipole Field



If two cylinders carrying equal and opposite current densities $+J$ and $-J$ are separated along the X-axis by a distance x_0, then the region of overlap carries no current, and may be replaced by a region of free space. For any point $\boldsymbol{z}_{in}$ in this "aperture", the complex field can be obtained from the expression for $\boldsymbol{B}_{in}(\boldsymbol{z})$ for a single cylinder:

$$\boldsymbol{B}(\boldsymbol{z}_{in}) = B_{in}^{(1)}(\boldsymbol{z}_1) + B_{in}^{(2)}(\boldsymbol{z}_2) = \left(\frac{\mu_0 J}{2}\right).\boldsymbol{z}_1^* - \left(\frac{\mu_0 J}{2}\right).\boldsymbol{z}_2^*$$

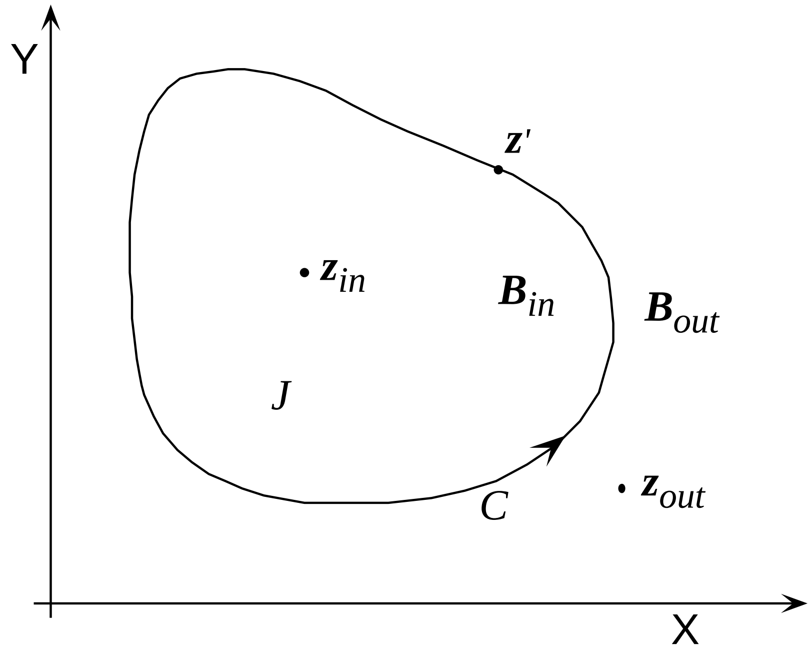
$$\boldsymbol{B}(\boldsymbol{z}_{in}) = \left(\frac{\mu_0 J}{2}\right).\left[\boldsymbol{z}_1^* - \boldsymbol{z}_2^*\right] = \left(\frac{\mu_0 J}{2}\right).\left[\left(\boldsymbol{z}_{in}^* + \frac{x_0}{2}\right) - \left(\boldsymbol{z}_{in}^* - \frac{x_0}{2}\right)\right] = \left(\frac{\mu_0 J x_0}{2}\right)$$

The complex potential is thus a constant throughout the "aperture". Since the complex potential is given by $B_y + iB_x$, we get in this case:

$$B_y = \left(\frac{\mu_0 J x_0}{2}\right); \quad B_x = 0$$

A pure dipole field is also produced by two overlapping cylinders with elliptical cross sections. Similarly, two ellipses placed at right angles produce a pure quadrupole field. In practice, the ends of the two current halves must be truncated, which gives rise to unacceptably large higher harmonics.

Solid Conductor of Arbitrary Shape: Integral Formula



Let us consider an infinitely long solid conductor of arbitrary cross section defined by the contour C. Inside the region of the conductor, the Maxwell's equations give:

$$(\nabla\times\mathbf{B})_z=\left(\frac{\partial B_y}{\partial x}\right)-\left(\frac{\partial B_x}{\partial y}\right)=\mu_0 J_z(x,y);\quad \nabla\cdot\mathbf{B}=\left(\frac{\partial Bx}{\partial x}\right)+\left(\frac{\partial B_y}{\partial y}\right)=0$$

For a constant current density, $J_z(x,y) = J$, it can be shown that the function:

$$\boldsymbol{F}(\boldsymbol{z})=B_y(x,y)+iB_x(x,y)-\left(\frac{\mu_0 J}{2}\right)\boldsymbol{z}^*=\boldsymbol{B}(\boldsymbol{z})-\left(\frac{\mu_0 J}{2}\right)\boldsymbol{z}^*$$

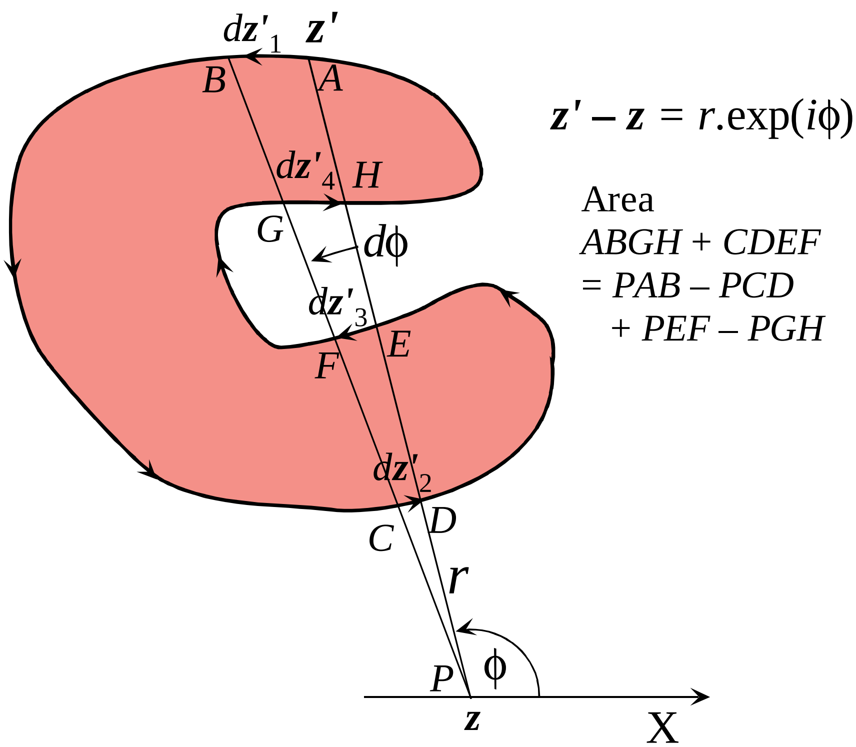
is an analytic function of z. This function can be evaluated as a contour integral over the boundary of the conductor as:

$$\boldsymbol{F}(\boldsymbol{z})=i\left(\frac{\mu_0 J}{4\pi}\right)\oint_C\frac{\boldsymbol{z}'^*}{\boldsymbol{z}'-\boldsymbol{z}}\,d\boldsymbol{z}'=\boldsymbol{B}_{in}(\boldsymbol{z})-\left(\frac{\mu_0 J}{2}\right)\boldsymbol{z}^*\quad\text{for } \boldsymbol{z}=\boldsymbol{z}_{in}$$

$\boldsymbol{B}_{out}(\boldsymbol{z})$ is the same as $\boldsymbol{F}(\boldsymbol{z})$ for $\boldsymbol{z}=\boldsymbol{z}_{out}$.

[Reference: R.A. Beth, *J. Appl. Phys*. **38**(12), 4689-92 (1967)]

Solid Conductor of Arbitrary Shape: Integral Formula



Field at $P(\mathbf{z})$ due to a triangular wedge *PAB* is given by:

$$d\mathbf{B}(\mathbf{z}) = -\left(\frac{m_0 J\, df}{2p}\right)\int_0^r \frac{r\,dr}{r\exp(if)} = -\left(\frac{m_0 J}{2p}\right) r\exp(-if)\,df = -\left(\frac{m_0 J}{2p}\right)(\mathbf{z}'-\mathbf{z})^*\,df$$

$$\mathbf{z}'-\mathbf{z} = r\exp(if);\quad d\mathbf{z}'/(\mathbf{z}'-\mathbf{z}) = (dr/r) + i\,df$$

$$\therefore df = \left(\frac{1}{2i}\right)\left(\frac{d\mathbf{z}'}{\mathbf{z}'-\mathbf{z}} - \frac{d\mathbf{z}'^*}{\mathbf{z}'^*-\mathbf{z}^*}\right)$$

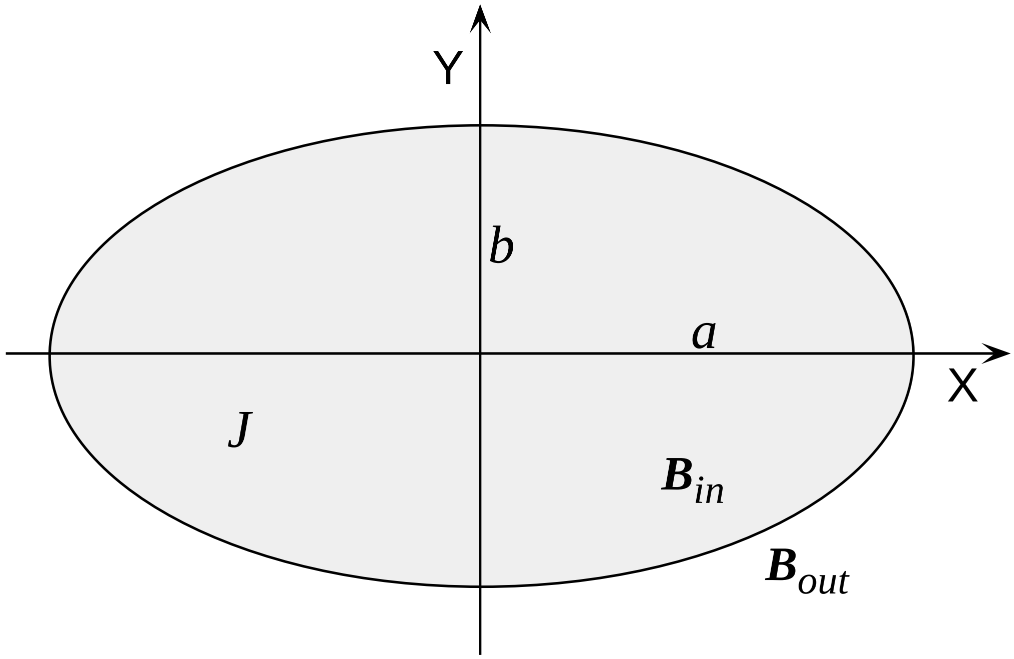
The total field at $P(\mathbf{z})$ due to the entire conductor is therefore,

$$\mathbf{B}(\mathbf{z}) = \left(\frac{m_0 J}{4p}\right) i\left[\oint \frac{(\mathbf{z}'-\mathbf{z})^* d\mathbf{z}'}{\mathbf{z}'-\mathbf{z}} - \oint d\mathbf{z}'^*\right] = \left(\frac{m_0 J}{4p}\right) i\oint \frac{(\mathbf{z}'-\mathbf{z})^* d\mathbf{z}'}{\mathbf{z}'-\mathbf{z}}$$

This applies to all points, whether inside or outside the conductor.

[Reference: R.A. Beth, *J. Appl. Phys.* **40**(12), 4782-6 (1969)]

Solid Elliptical Conductor with Uniform Current Density



The field inside and outside the elliptical conductor shown in the figure can be obtained using the integral formula for 2-D fields [R.A. Beth, *J. Appl. Phys*. **38**(12), 4689-92 (1967)]. The result is:

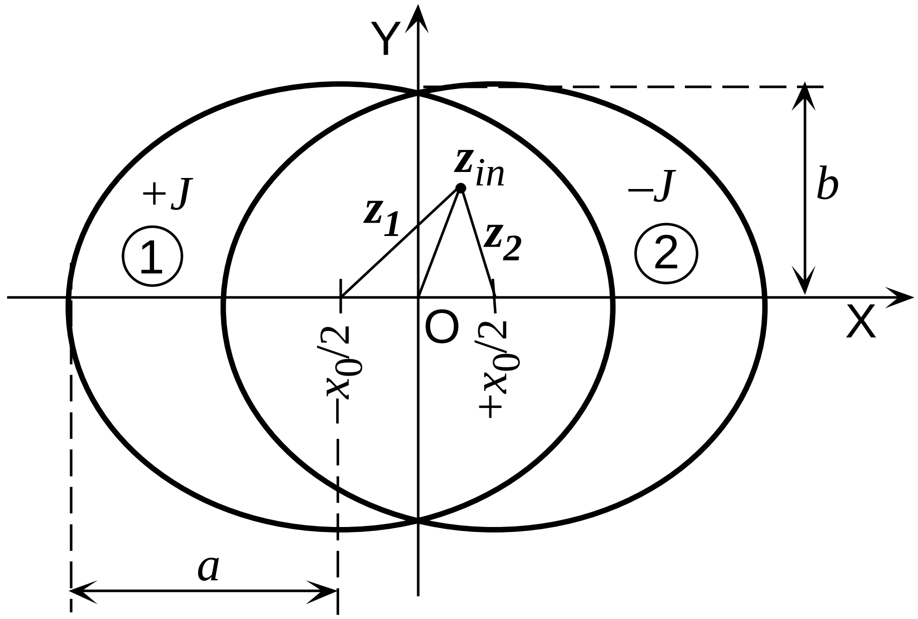
$$\boldsymbol{B}_{in}(\boldsymbol{z}) = \frac{\mu_o J}{(a+b)}[bx - iay]; \quad \boldsymbol{B}_{out}(\boldsymbol{z}) = \left(\frac{\mu_o J}{2}\right)\left[\frac{2ab}{\boldsymbol{z} + \sqrt{\boldsymbol{z}^2 - (a^2 - b^2)}}\right]$$

For a circular conductor, $b = a$, and these expressions reduce to:

$$\boldsymbol{B}_{in}(\boldsymbol{z}) = \left(\frac{\mu_o J}{2}\right)\boldsymbol{z}^*; \quad \boldsymbol{B}_{out}(\boldsymbol{z}) = \left(\frac{\mu_o J}{2}\right)\left(\frac{a^2}{\boldsymbol{z}}\right) \qquad \text{CIRCULAR CONDUCTOR}$$

Using the expression for $\boldsymbol{B}_{in}(\boldsymbol{z})$ it can be shown that two overlapping ellipses can be used to produce a pure dipole, or a pure quadrupole field.

Pure Dipole Field Using Two Elliptical Conductors



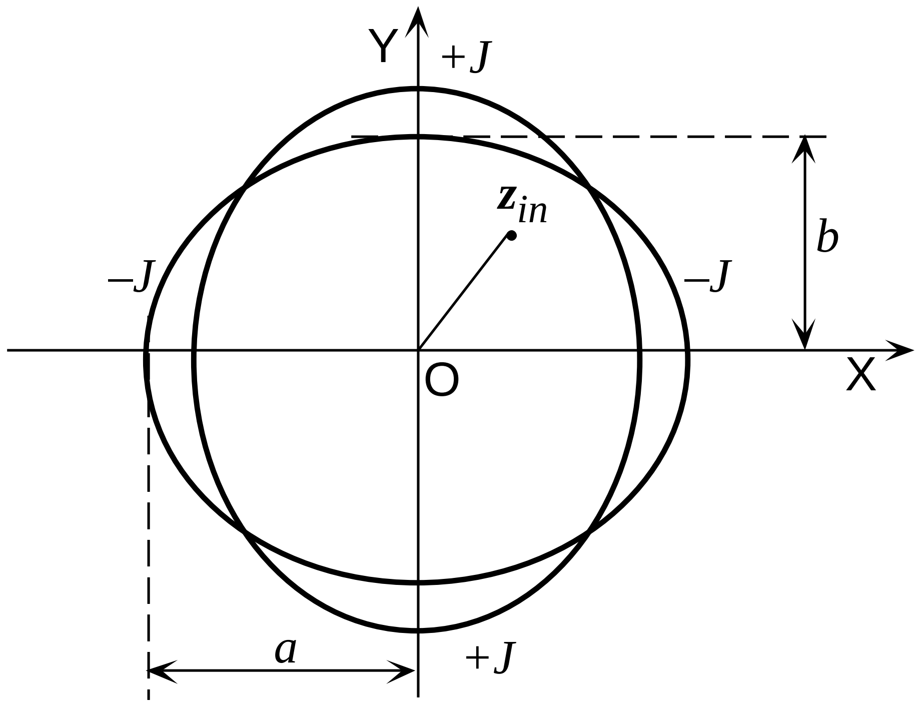
The field inside the "aperture" formed by two intersecting ellipses carrying equal and opposite current densities is given by:

$$\boldsymbol{B}_{in}(\boldsymbol{z}) = \frac{\mu_o J}{(a+b)}\left[b\left(x+\frac{x_0}{2}\right) - iay - b\left(x-\frac{x_0}{2}\right) + iay\right] = \frac{\mu_o J b x_0}{(a+b)} = \text{ const.}$$

$$B_y(\boldsymbol{z}) = \frac{\mu_o J b x_0}{(a+b)} = \text{ const.}; \quad B_x(\boldsymbol{z}) = 0$$

This represents a pure dipole field. In practice, the sharp corners in the current blocks must be truncated. This gives rise to unacceptable higher harmonics with this simple approach.

Pure Quadrupole Field Using Two Elliptical Conductors



The field inside the "aperture" formed by the two ellipses carrying equal and opposite current densities and intersecting at right angles is given by:

$$\boldsymbol{B}_{in}(\boldsymbol{z}) = \frac{\mu_o J}{(a+b)}\left[-bx + iay + ax - iby\right] = \frac{\mu_o J(a-b)}{(a+b)}(x+iy)$$

$$B_y(\boldsymbol{z}) = \frac{\mu_o J(a-b)}{(a+b)}\,x\,; \quad B_x(\boldsymbol{z}) = \frac{\mu_o J(a-b)}{(a+b)}\,y$$

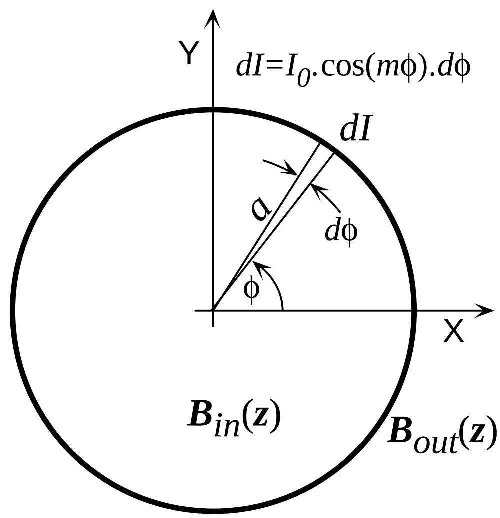
This represents a pure quadrupole field. The gradient is given by:

$$G = \left(\frac{\partial B_y}{\partial x}\right) = \left(\frac{\partial B_x}{\partial y}\right) = \frac{\mu_o J(a-b)}{(a+b)} = \text{ constant.}$$

Generating a Pure 2*m*-Pole Field: The Cos(*m*θ) Distribution

Let us consider a cylindrical current sheet of radius a composed of thin filaments of current flowing in the Z-dirction only.
The current density at any angle ϕ is assumed to be proportional to $\cos(m\phi)$ where m is a non-zero positive integer

$$dI = I_0 \cos(m\phi).d\phi$$



The complex field at any point can be obtained by integrating the contributions from small elements of width $d\phi$, as shown in the figure.

$$\boldsymbol{B}_{in}(\boldsymbol{z}) = -\left(\frac{\mu_0 I_0}{2\pi a}\right)\int_0^{2\pi} \sum_{n=1}^{\infty}\left(\frac{\boldsymbol{z}}{a}\right)^{n-1} \exp(-in\phi)\cos(m\phi)d\phi$$

$$\int_0^{2\pi}\cos(n\phi)\cos(m\phi)d\phi = \pi\delta_{mn}; \quad \int_0^{2\pi}\sin(n\phi)\cos(m\phi)d\phi = 0$$

$$\therefore \boldsymbol{B}_{in}(\boldsymbol{z}) = -\left(\frac{\mu_0 I_0}{2a}\right)\left(\frac{\boldsymbol{z}}{a}\right)^{m-1}$$

Accordingly, the radial and the azimuthal components of the field are:

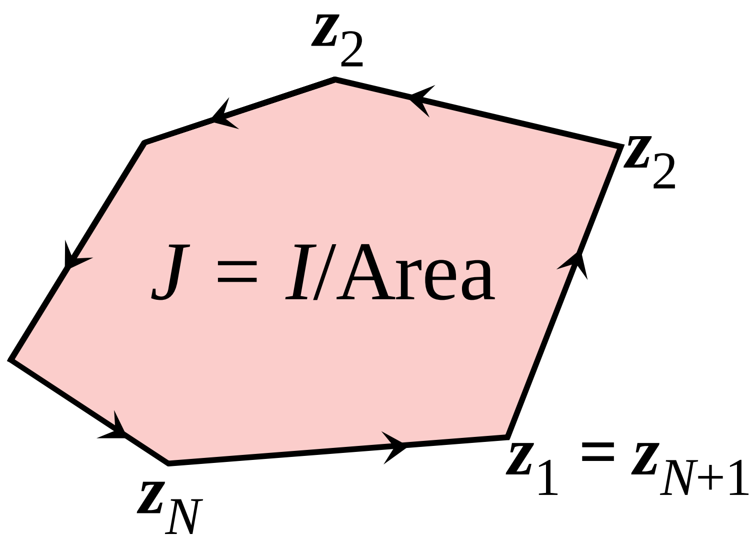
$$B_r(r,\theta) = \left[-\left(\frac{\mu_0 I_0}{2a}\right)\left(\frac{R_{ref}}{a}\right)^{m-1}\right]\left(\frac{r}{R_{ref}}\right)^{m-1}\sin(m\theta); \quad B_\theta(r,\theta) = \left[-\left(\frac{\mu_0 I_0}{2a}\right)\left(\frac{R_{ref}}{a}\right)^{m-1}\right]\left(\frac{r}{R_{ref}}\right)^{m-1}\cos(m\theta)$$

These components represent a pure 2m-pole field. Sometimes, it is important to know the field outside the current shell, which is given by:

$$\boldsymbol{B}_{out}(\boldsymbol{z}) = \left(\frac{\mu_0 I_0}{2\pi \boldsymbol{z}}\right)\int_0^{2\pi}\left[1+\sum_{n=1}^{\infty}\exp(in\phi)\left(\frac{a}{\boldsymbol{z}}\right)^n\right]\cos(m\phi)d\phi = \left(\frac{\mu_0 I_0}{2a}\right)\left(\frac{a}{\boldsymbol{z}}\right)^{m+1}$$

The field thus falls off as $(1/r)^{m+1}$ outside the current shell.

Field Due to a Conductor of Polygonal Cross Section



$$\boldsymbol{B}_{out}(\mathbf{z}) = \left(\frac{m_0 J}{4p}\right) i \oint \frac{\mathbf{z}'^*}{\mathbf{z}' - \mathbf{z}}\, d\mathbf{z}' = \left(\frac{m_0 J}{4p}\right) \sum_{j=1}^{N} I_j(\mathbf{z})$$

$$\boldsymbol{B}_{in}(\mathbf{z}) = \left(\frac{m_0 J}{4p}\right) i \oint \frac{\mathbf{z}'^*}{\mathbf{z}' - \mathbf{z}}\, d\mathbf{z}' + \left(\frac{m_0 J}{2}\right) \mathbf{z}^* = \left(\frac{m_0 J}{4p}\right) \left[2p\, \mathbf{z}^* + \sum_{j=1}^{N} I_j(\mathbf{z}) \right]$$

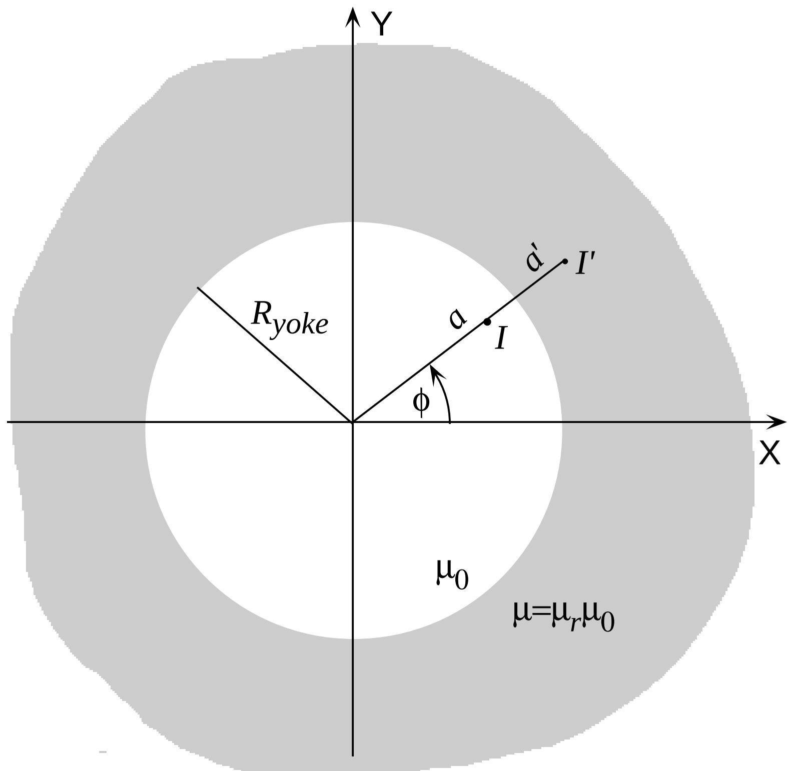
where, $$I_j(\mathbf{z}) = i \left[\left(\mathbf{z}_{j+1}^* - \mathbf{z}_j^* \right) + \frac{\mathbf{z}_j^* \left(\mathbf{z}_{j+1} - \mathbf{z} \right) - \mathbf{z}_{j+1}^* \left(\mathbf{z}_j - \mathbf{z} \right)}{\mathbf{z}_{j+1} - \mathbf{z}_j} \ln \left(\frac{\mathbf{z}_{j+1} - \mathbf{z}}{\mathbf{z}_j - \mathbf{z}} \right) \right]$$

is the contribution to the contour integral from the *j*-th side of the polygon. The first term in $I_j(\mathbf{z})$ gives zero contribution when summed over all the sides, and can be omitted in the expression for the field. The cross-sectional area (needed to evaluate the current density) is given by:

$$A = \frac{1}{2} \sum_{j=1}^{N} (x_j\, y_{j+1} - x_{j+1}\, y_j)$$

A conductor of arbitrary cross-section can also be approximated by a polygon with a sufficiently large number of sides, and the above equations can be used to evaluate field from such a conductor.

A Current Filament Inside a Cylindrical Iron Yoke



By the method of images, the effect of the iron yoke on the field inside the "aperture" of the yoke can be described by replacing it with an image current given by:

$$a' = \frac{R_{yoke}^2}{a}; \quad I' = \left(\frac{\mu_r - 1}{\mu_r + 1}\right) I; \quad \phi' = \phi$$

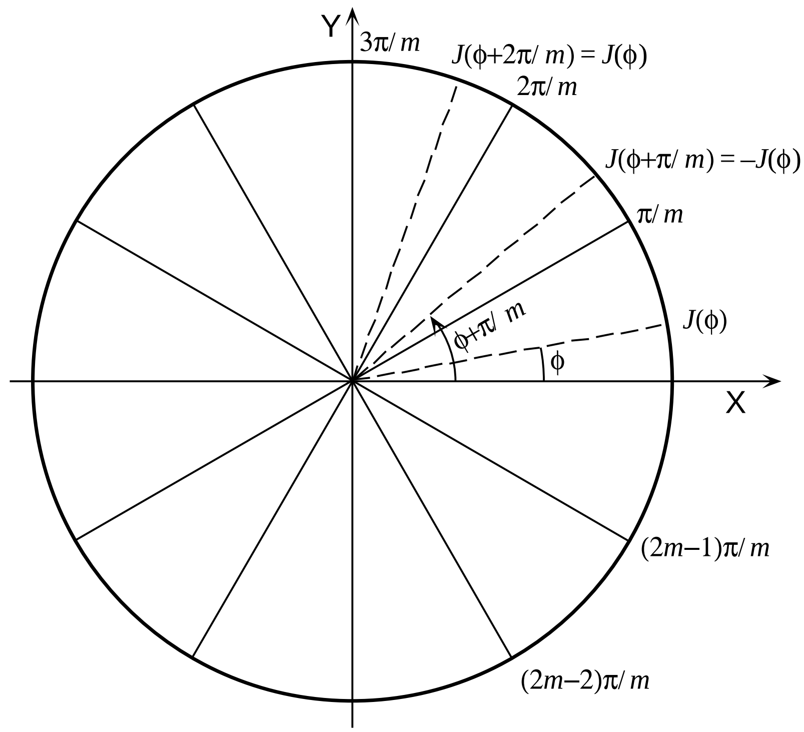
The coefficient of the $2n$-pole term in the multipole expansion of the field is:

$$C(n)\exp(-in\alpha_n) = -\left(\frac{\mu_0 I}{2\pi a}\right)\left(\frac{R_{ref}}{a}\right)^{n-1}\left[1 + \left(\frac{\mu_r - 1}{\mu_r + 1}\right)\left(\frac{a}{R_{yoke}}\right)^{2n}\right].\exp(-in\phi)$$

The presence of yoke results in an increase of field in the aperture. Since $a < R_{yoke}$, the enhancement in field reduces with the order of the multipole, n.

Allowed Harmonics in a "2*m*-pole" Magnet

The current distribution in a "2*m*-pole" magnet has a *m*-fold rotational symmetry. In addition, the current distribution is antisymmetric under a rotation by (π/m) radians, as shown in the figure. Certain harmonics in the multipole expansion vanish under these conditions, leading to only a selected group of harmonics being allowed for the magnet.



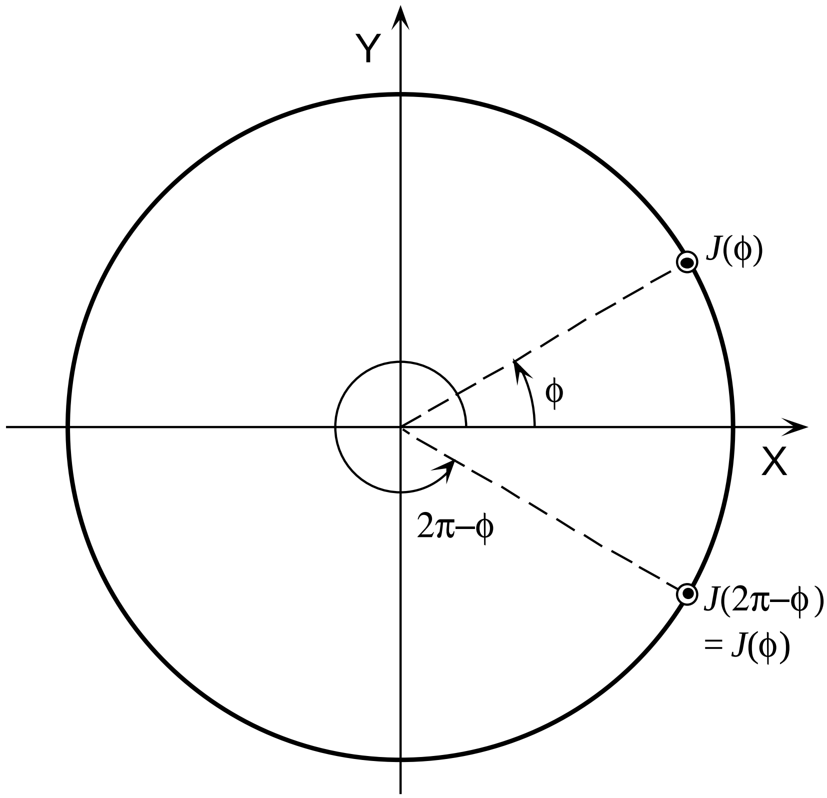
$$C(n)\exp(in\alpha_n) \propto \int_0^{2\pi} J(\phi)e^{in\phi}d\phi = \int_0^{\pi/m} J(\phi)e^{in\phi}d\phi + \int_{\pi/m}^{2\pi/m} J(\phi)e^{in\phi}d\phi + \cdots$$

$$C(n)\exp(in\alpha_n) \propto \int_0^{\pi/m} J(\phi)e^{in\phi}[1-e^{in\pi/m}+e^{2in\pi/m}+\cdots -e^{i(2m-1)\pi/m}]d\phi$$

$$\propto \left[\int_0^{\pi/m} J(\phi)e^{in\phi}d\phi\right]\cdot\left[\frac{1-e^{2in\pi}}{1+e^{in\pi/m}}\right]$$

This vanishes unless n is an odd multiple of m. For a dipole magnet (m=1), only terms with n = 1,3,5,... are allowed. For a quadrupole magnet (m=2), terms with n = 2,6,10,... are allowed and so on.

“Top-Bottom” Symmetry in the Current Distribution



The current density satisfies $J(\phi) = J(2\pi-\phi)$, as shown in the figure. The normal and skew multipoles are given by:

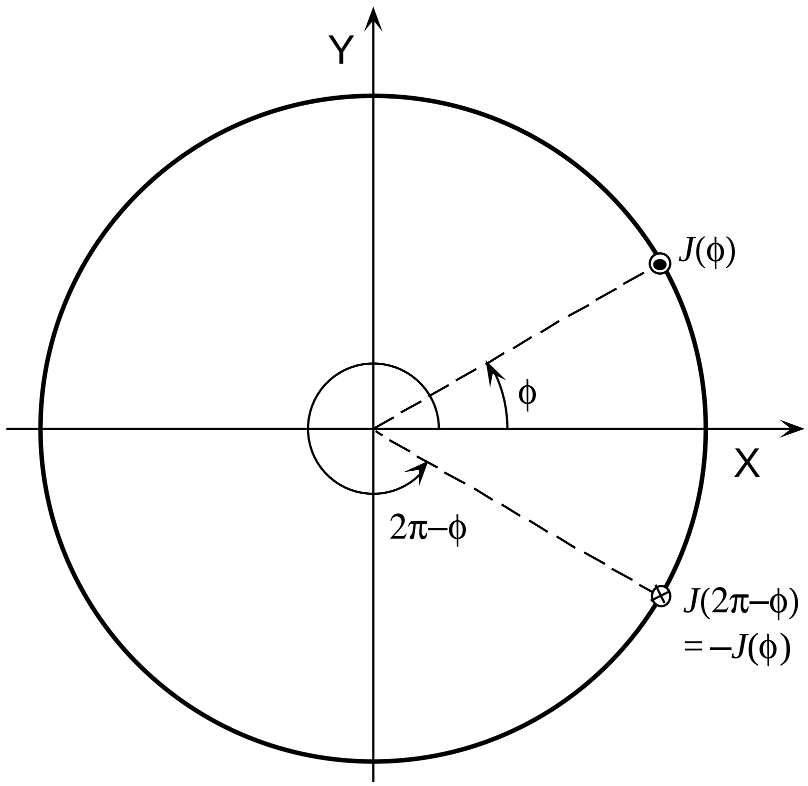
$$C(n)\exp(in\alpha_n) \propto \int_0^{2\pi} J(\phi)e^{in\phi}d\phi = \int_0^{\pi} J(\phi)\left[e^{in\phi} + e^{in(2\pi-\phi)}\right]d\phi = \int_0^{\pi} J(\phi)\left[e^{in\phi} + e^{-in\phi}\right]d\phi$$

$$\therefore C(n)\exp(in\alpha_n) \propto \int_0^{\pi} J(\phi)\cos(n\phi)d\phi$$

The result of integration has no imaginary part in this case. This implies that all the skew terms vanish as a result of the “top-bottom” symmetry in the current distribution.

TOP-BOTTOM SYMMETRY IN CURRENT DENSITY
Þ ALL SKEW TERMS ARE ZERO.

“Top-Bottom” Anti-Symmetry in the Current Distribution



The current density satisfies $J(\phi) = -J(2\pi-\phi)$, as shown in the figure. The normal and skew multipoles are given by:

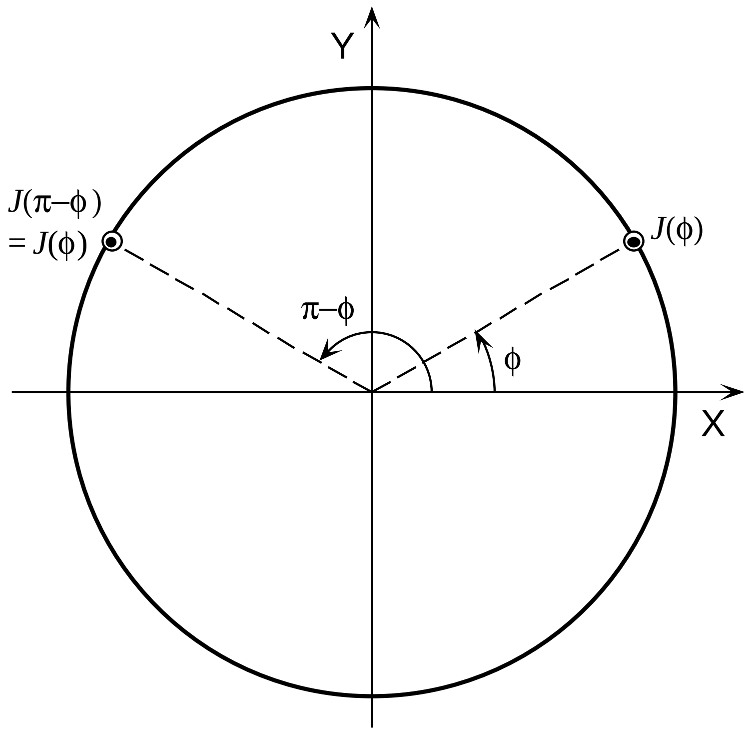
$$C(n)\exp(in\alpha_n) \propto \int_0^{2\pi} J(\phi)e^{in\phi}d\phi = \int_0^{\pi} J(\phi)\left[e^{in\phi} - e^{in(2\pi-\phi)}\right]d\phi = \int_0^{\pi} J(\phi)\left[e^{in\phi} - e^{-in\phi}\right]d\phi$$

$$\therefore C(n)\exp(in\alpha_n) \propto i\int_0^{\pi} J(\phi)\sin(n\phi)d\phi$$

The result of integration has no real part in this case. This implies that all the normal terms vanish as a result of the “top-bottom” anti-symmetry in the current distribution.

TOP-BOTTOM ANTI-SYMMETRY IN CURRENT DENSITY
Þ ALL NORMAL TERMS ARE ZERO.

"Left-Right" Symmetry in the Current Distribution



The current density satisfies $J(\phi) = J(\pi-\phi)$, as shown in the figure. The normal and skew multipoles are given by:

$$C(n)\exp(in\alpha_n) \propto \int_0^{2\pi} J(\phi) e^{in\phi} d\phi = \int_{-\pi/2}^{\pi/2} J(\phi)\left[e^{in\phi} + e^{in(\pi-\phi)}\right] d\phi = \int_{-\pi/2}^{\pi/2} J(\phi)\left[e^{in\phi} + (-1)^n e^{-in\phi}\right] d\phi$$

$$C(n)\exp(in\alpha_n) \propto i \int_{-\pi/2}^{\pi/2} J(\phi)\sin(n\phi) d\phi \quad \text{for ODD } n$$

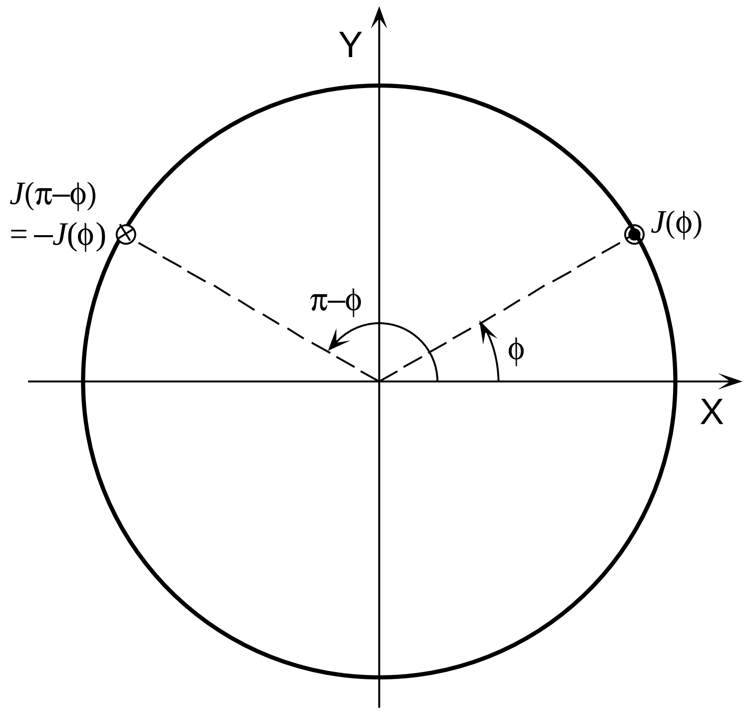
$$C(n)\exp(in\alpha_n) \propto \int_{-\pi/2}^{\pi/2} J(\phi)\cos(n\phi) d\phi \quad \text{for EVEN } n$$

The result of integration has no real part for odd multipoles and has no imaginary part for even multipoles. This implies that all the odd normal terms (such as normal dipole, normal sextupole, etc.) and all the even skew terms (such as skew quadrupole, skew octupole, etc.) vanish as a result of the "left-right" symmetry in the current distribution.

LEFT-RIGHT SYMMETRY IN CURRENT DENSITY

Þ ODD NORMAL TERMS ARE ZERO [2(2*k*+1)-POLE]
EVEN SKEW TERMS ARE ZERO [2(2*k*)-POLE]

"Left-Right" Anti-Symmetry in the Current Distribution



The current density satisfies $J(\phi) = -J(\pi-\phi)$, as shown in the figure. The normal and skew multipoles are given by:

$$C(n)\exp(in\alpha_n) \propto \int_0^{2\pi} J(\phi)e^{in\phi}d\phi = \int_{-\pi/2}^{\pi/2} J(\phi)\left[e^{in\phi} - e^{in(\pi-\phi)}\right]d\phi = \int_{-\pi/2}^{\pi/2} J(\phi)\left[e^{in\phi} - (-1)^n e^{-in\phi}\right]d\phi$$

$$C(n)\exp(in\alpha_n) \propto i\int_{-\pi/2}^{\pi/2} J(\phi)\sin(n\phi)d\phi \quad \text{for EVEN } n$$

$$C(n)\exp(in\alpha_n) \propto \int_{-\pi/2}^{\pi/2} J(\phi)\cos(n\phi)d\phi \quad \text{for ODD } n$$

The result of integration has no real part for even multipoles and has no imaginary part for odd multipoles. This implies that all the even normal terms (such as normal quadrupole, normal octupole, etc.) and all the odd skew terms (such as skew dipole, skew sextupole, etc.) vanish as a result of the "left-right" anti-symmetry in the current distribution.

LEFT-RIGHT ANTI-SYMMETRY IN CURRENT DENSITY

Þ EVEN NORMAL TERMS ARE ZERO [2(2*k*)-POLE]
ODD SKEW TERMS ARE ZERO [2(2*k*+1)-POLE]

Two-Dimensional Behaviour of Integral Field

In magnets of a finite length, the two dimensional representation of the field is valid only in the body of the magnet, sufficiently away from the ends. In regions near the ends of the magnet, the field is three dimensional and the usual multipole expansion is no longer valid. However, for most practical purposes, one is interested in the **integral** of the field (and its various derivatives) over the length of the magnet. Also, a measuring coil of finite length only measures integral of the field over its length. It can be shown that the integral field essentially behaves as a two dimensional field provided the integration is carried out over an appropriate region.

In general, for three dimensional fields, the scalar potential satisfies Laplace's equation:

$$\nabla^2\Phi_m(x,y,z) = \left[\frac{\partial^2\Phi_m}{\partial x^2} + \frac{\partial^2\Phi_m}{\partial y^2} + \frac{\partial^2\Phi_m}{\partial z^2}\right] = 0$$

Integrating along the Z-axis from Z_1 to Z_2, we get:

$$\int_{Z_1}^{Z_2}\left[\frac{\partial^2\Phi_m}{\partial x^2} + \frac{\partial^2\Phi_m}{\partial y^2} + \frac{\partial^2\Phi_m}{\partial z^2}\right]dz = \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]\int_{Z_1}^{Z_2}\Phi_m(x,y,z)\,dz + \int_{Z_1}^{Z_2}\frac{\partial^2\Phi_m}{\partial z^2}\,dz = 0$$

We define the z-integrated scalar potential as $\overline{\Phi}_m(x,y) = \int_{Z_1}^{Z_2}\Phi_m(x,y,z)\,dz$.

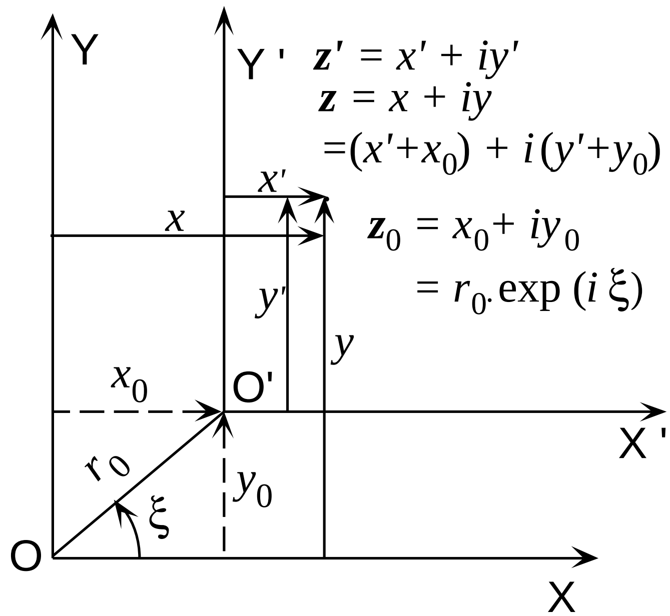
$$\left[\frac{\partial^2\overline{\Phi}_m}{\partial x^2} + \frac{\partial^2\overline{\Phi}_m}{\partial y^2}\right] = -\left[\frac{\partial\Phi_m}{\partial z}\right]_{Z_1}^{Z_2} = B_z(x,y,Z_2) - B_z(x,y,Z_1)$$

If the region of integration is so chosen that the Z-component of the field is zero at the boundaries of this region, then the right hand side vanishes and the average scalar potential satisfies the two dimensional Laplace's equation. For example, the points Z_1 and Z_2 could both be chosen well outside the magnet on opposite ends. Alternatively, one could choose Z_1 well outside the magnet and Z_2 well inside the magnet, where the field is again two dimensional.

Transformation of Field Parameters under Displacement of Axes

The expansion parameters $[C(n), \alpha_n]$ or $[B_n, A_n]$ depend on the choice of the reference frame.

Let us consider a frame X$'$-Y$'$ which is displaced from a frame X-Y by $\boldsymbol{z}_0$, as shown in the figure. The field parameters are denoted by $[C(n), \alpha_n]$ or $[B_n, A_n]$ in the X-Y frame and by $[C'(n), \alpha'_n]$ or $[B'_n, A'_n]$ in the X$'$-Y$'$ frame.

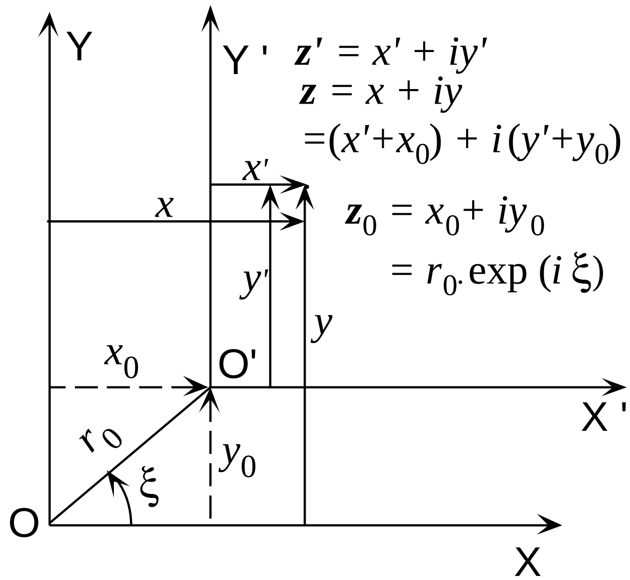


Since the Cartesian components of **B** are the same in the two reference frames, we have (with the "US Convention"):

$$\begin{aligned}
\boldsymbol{B}(\boldsymbol{z}') &= B_{y'} + iB_{x'} = B_y + iB_x \\
&= \sum_{k=0}^{\infty}(B_k + iA_k)\left(\frac{\boldsymbol{z}}{R_{ref}}\right)^k = \sum_{k=0}^{\infty}(B_k + iA_k)\left(\frac{\boldsymbol{z}' + \boldsymbol{z}_0}{R_{ref}}\right)^k \\
&= \sum_{k=0}^{\infty}(B_k + iA_k)\sum_{n=0}^{k}\frac{k!}{n!(k-n)!}\left(\frac{\boldsymbol{z}'}{R_{ref}}\right)^n\left(\frac{\boldsymbol{z}_0}{R_{ref}}\right)^{k-n} \\
&= \sum_{n=0}^{\infty}\left[\sum_{k=n}^{\infty}(B_k + iA_k)\frac{k!}{n!(k-n)!}\left(\frac{\boldsymbol{z}_0}{R_{ref}}\right)^{k-n}\right]\left(\frac{\boldsymbol{z}'}{R_{ref}}\right)^n \\
&= \sum_{n=0}^{\infty}(B'_n + iA'_n)\left(\frac{\boldsymbol{z}'}{R_{ref}}\right)^n
\end{aligned}$$

where we have used the identity $\displaystyle\sum_{k=0}^{\infty}\sum_{n=0}^{k} t_{kn} = \sum_{n=0}^{\infty}\sum_{k=n}^{\infty} t_{kn}$

Transformation of Field Parameters under Displacement of Axes



$$(B_n' + iA_n') = \sum_{k=n}^{\infty} (B_k + iA_k) \left[\frac{k!}{n!(k-n)!} \right] \left(\frac{x_0 + iy_0}{R_{ref}} \right)^{k-n} \quad ; \quad n \geq 0$$

US NOTATION

In the European notation, the indices of the B's and the A's in the above equation are increased by one. Hence, we may rewrite the above equation as:

$$(B_n' + iA_n') = \sum_{k=n}^{\infty} (B_k + iA_k) \left[\frac{(k-1)!}{(n-1)!(k-n)!} \right] \left(\frac{x_0 + iy_0}{R_{ref}} \right)^{k-n} \quad ; \quad n \geq 1$$

EUROPEAN NOTATION

The transformation for the Amplitude and phase of the $2n$-pole term is:

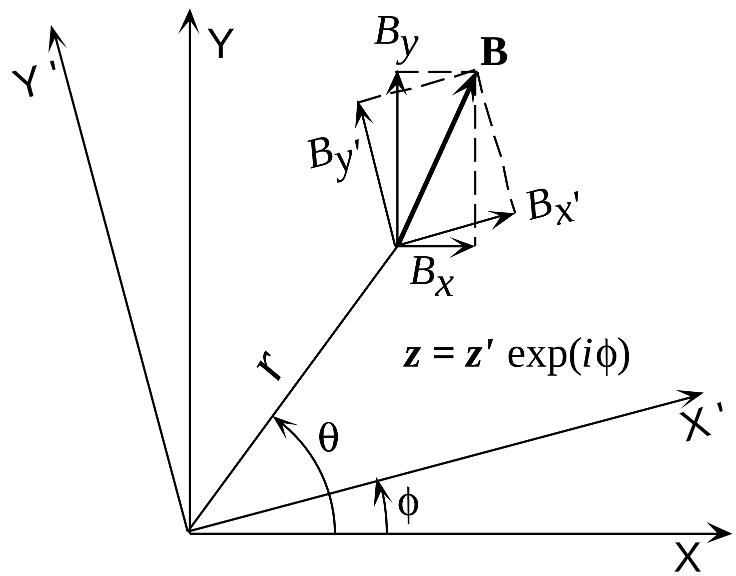
$$C'(n) \exp(-in\alpha_n') = \sum_{k=n}^{\infty} [C(k) \exp(-ik\alpha_k)] \left[\frac{(k-1)!}{(n-1)!(k-n)!} \right] \left(\frac{x_0 + iy_0}{R_{ref}} \right)^{k-n} \quad ; \quad n \geq 1$$

Coefficients of any particular order in the displaced frame are given by a combination of ALL THE TERMS OF EQUAL OR HIGHER ORDER in the undisplaced frame. This effect is referred to as the FEED DOWN of harmonics.

Transformation of Field Parameters under Rotation of Axes

X-Y : Original Frame
X′-Y′ : Rotated Frame
ϕ = Rotation angle
$\boldsymbol{z} = r.\exp(i\theta)$
$\boldsymbol{z'} = r.\exp(i\theta')$
$\theta = \theta' + \phi$

$\boldsymbol{z} = \boldsymbol{z'} \exp(i\phi)$



$$B_{x'} = B_x \cos\phi + B_y \sin\phi \;; \quad B_{y'} = -B_x \sin\phi + B_y \cos\phi$$

$$\boldsymbol{B}(\boldsymbol{z'}) = B_{y'} + iB_{x'} = (B_y + iB_x)\exp(i\phi)$$

$$= \sum_{n=0}^{\infty} [B_n + iA_n] \left(\frac{\boldsymbol{z}}{R_{ref}}\right)^n \exp(i\phi)$$

$$= \sum_{n=0}^{\infty} [B_n + iA_n] \left(\frac{\boldsymbol{z'}}{R_{ref}}\right)^n \exp[i(n+1)\phi] = \sum_{n=0}^{\infty} [B'_n + iA'_n] \left(\frac{\boldsymbol{z'}}{R_{ref}}\right)^n$$

$$(B'_n + iA'_n) = (B_n + iA_n)\exp[i(n+1)\phi] \;; \quad n \ge 0$$ “US NOTATION”

$$(B'_n + iA'_n) = (B_n + iA_n)\exp(in\phi) \;; \quad n \ge 1$$ “EUROPEAN NOTATION”

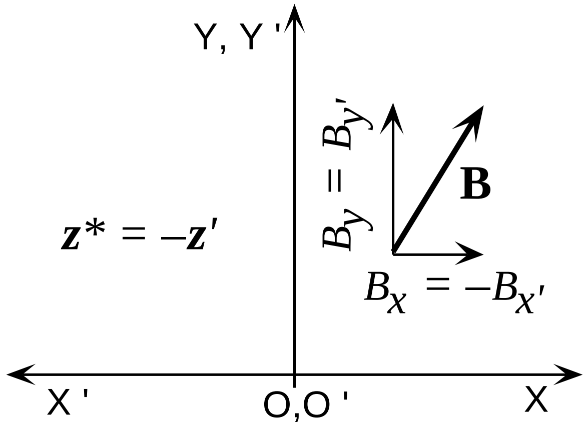
$$C'(n)\exp(-in\alpha'_n) = C(n)\exp(-in\alpha_n)\exp(in\phi)$$

$$\text{or,} \quad C'(n) = C(n); \quad \alpha'_n = \alpha_n - \phi; \quad n \ge 1$$

A rotation of axes causes NO FEED DOWN of harmonics, but causes mixing of Normal and Skew components of a given harmonic.

Transformation of Field Parameters under Reflection of Axes

If a magnet is viewed from an end which is opposite to the end from which the field parameters are measured, then appropriate transformations must be applied to the field parameters. This situation corresponds to the figure here, where the X′ axis points away from the X-axis and the Y′-axis coincides with the Y-axis.



$$z^* = x - iy = -x' - iy' = -z'$$

$$B_{y'} + iB_{x'} = B_y - iB_x = (B_y + iB_x)^*$$

$$= \sum_{n=0}^{\infty} (B_n + iA_n)^* \left(\frac{z^*}{R_{ref}}\right)^n$$

$$= \sum_{n=0}^{\infty} (B_n - iA_n)(-1)^n \left(\frac{z'}{R_{ref}}\right)^n = \sum_{n=0}^{\infty} (B'_n + iA'_n) \left(\frac{z'}{R_{ref}}\right)^n$$

$$B'_n = (-1)^n B_n \; ; \quad A'_n = (-1)^{n+1} A_n \; ; \quad n \geq 0 \qquad \text{US NOTATION}$$

$$B'_n = (-1)^{n+1} B_n \; ; \quad A'_n = (-1)^n A_n \; ; \quad n \geq 1 \qquad \text{EUROPEAN NOTATION}$$

$$C'(k) = C(k); \quad \alpha'_{2k-1} = -\alpha_{2k-1}; \quad \alpha'_{2k} = \left(\frac{\pi}{2k}\right) - \alpha_{2k}; \quad k \geq 1$$

There is NO Feed Down of harmonics, or mixing of Normal and Skew components under this transformation. Simply, the signs of alternate normal and skew components are changed. The terms that change sign are the skew dipole, the normal quadrupole, the skew sextupole, the normal octupole, and so on.